Preservation theorems and restricted consistency statements in bounded arithmetic*

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Abstract

In this article we will define and study a new restricted consistency notion $\mathrm{RCon}^*(T^j_2)$ for bounded arithmetic theories T^j_2. It will be the strongest $\forall\Pi^b_1$-statement over S^1_2 provable in T^j_2, similar to $Con(G_i)$ in [13] or $RCon(T^i_1)$ in [14]. The advantage of our notion over the others is that $\mathrm{RCon}^*(T^j_2)$ can directly be used to construct models of T^j_2. We apply this by proving preservation theorems for theories of bounded arithmetic of the following well-known kind: The $\forall\Pi^b_1$-separation of bounded arithmetic theories S^i_2 from T^j_2 $(1 \le i \le j)$ is equivalent to the existence of a model of S^i_2 which does not have a Δ^b_0-elementary extension to a model of T^j_2.

More specific, let $M \models \Omega^{\mathrm{nst}}_1$ denote that there is a nonstandard element c in M such that the function $n \mapsto 2^{\log(n)^c}$ is total in M. Let $BL\Sigma^b_1$ be the bounded collection schema for Σ^b_1-formulas. We obtain the following preservation results: The $\forall\Pi^b_1$-separation of S^i_2 from T^j_2 $(1 \le i \le j)$ is equivalent to the existence of

1. a model of $S^i_2 + \Omega^{\mathrm{nst}}_1$ which is 1^b-closed w.r.t. T^j_2,
2. a countable model of $S^i_2 + BL\Sigma^b_1$ without weak end extensions to models of T^j_2.

This article is a contribution to the investigation of the influence of consistency notions to the finitely axiomatization question of bounded arithmetic. The usual notion of consistency is too strong to serve as a separating sentence for bounded arithmetic theories because $S_2 \nvdash Con_{S^{-1}_2}$, c.f. [21], where S^{-1}_2 is the induction-free fragment of bounded arithmetic S_2. Also the weaker consistency statement $BDCon$, which refers to proofs that use only bounded formulas, still is too strong: Buss in [6] proved that $S^{i+1}_2 \vdash BDCon_{S^i_2}$ holds for at most one i, and later Pudlák showed in [18] that $S_2 \nvdash BDCon_{S^1_2}$, hence only $S_2 \vdash BDCon_{S^0_2}$ remains to be possible. On the other hand I have been able to

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show in [4] that S^1_2 can prove the consistency of equational theories which base only on the recursive definition of the underlying function symbols. In particular $S^1_2 \vdash Con(S_2^{-\infty})$, where $S_2^{-\infty}$ is the equational theory based on the recursive definition of the function symbols of bounded arithmetic. This result disproves a plausible conjecture of TAKEUTI ([15] p.5 problem 9.). It gives hope that consistency statements can lead to a negative answer of the finitely axiomatization question of bounded arithmetic.

The focus of this paper are new restricted consistency statements for theories of bounded arithmetic and applications of them for proving preservation theorems for theories of bounded arithmetic in the manner of the following well-known one.[1] Let $\mathcal{L}$ be a first order language, $S \subseteq T$ be $\mathcal{L}$-theories and Δ a class of $\mathcal{L}$-formulas which is closed under conjunction and negation. With $\forall\Delta$ we denote the universal closure of (all formulas in) Δ.

Fact 1. *S is $\forall\Delta$-separated from T if and only if there is a model M of S which cannot be extended Δ-elementarily to a model of T.*

Proof ideas. The direction from left to right follows directly from the assumptions using the upwards persistence of $\exists\Delta$-formulas w.r.t. Δ-elementary extensions.

For the direction from right to left let M be a model of S which cannot be extended Δ-elementarily to a model of T. Then T plus the Δ-diagram of M is inconsistent. Using compactness (and the closure of Δ under conjunction) we obtain some $\varphi(\vec{a})$ in the Δ-diagram of M such that $T + \varphi(\vec{a})$ is inconsistent, hence $T \vdash \neg\varphi(\vec{a})$. Applying the lemma of new constants we obtain $T \vdash \forall\vec{x}\neg\varphi(\vec{x})$. On the other hand $M \models \varphi(\vec{a})$, hence $M \models \exists\vec{x}\varphi(\vec{x})$. Thus $S \neq_{\forall\Delta} T$. □

Introducing bounded arithmetic

Before we explain which restricted consistency statements we will consider and which preservation theorems will be proved by them let us briefly introduce bounded arithmetic. Bounded arithmetic is intended to characterize low complexity computability, i.e. the polynomial hierarchy. Every primitive recursive function is provable total in $I\Sigma_1$, hence $I\Sigma_1$ is much stronger than bounded arithmetic. By PARIKH's Theorem ([16], or see [6], p.83, Theorem 11) the provable total functions of $I\Delta_0$ (in the language $\mathcal{L}_{PA}$ of PEANO arithmetic) are bounded by polynomials. Hence $I\Delta_0(\mathcal{L}_{PA})$ is weaker than bounded arithmetic. Furthermore, only a constant number of elements $\leq n$ can be coded in a sequence $s = n^{O(1)}$: If we try to code l elements $\leq n$ in s we get

$$s = \underbrace{\overbrace{\rule{2em}{0.4pt}}^{\log n \text{ bits}} \ldots \overbrace{\rule{2em}{0.4pt}}^{\log n \text{ bits}}}_{l \text{ times}}$$

hence s consists of $l \cdot \log n$ bits, hence $s \approx n^l$. Thus, metamathematical arguments are in general not formalizable in $I\Delta_0(\mathcal{L}_{PA})$.

Allowing $l = m$ many elements $\leq n$ would result in an exponential growth rate, again too strong.

[1] I am grateful to the anonymous referee of a predecessor of this article for drawing my attention to general preservation theorems including Fact 1; and to Alex Wilkie for pointing out the simple proof of Fact 1 to me

As argued in [16] the right growth rate is obtained by allowing $l = \log m$ elements $\leq n$ to be coded into one sequence. Then

$$s = n^{\log m} \approx 2^{|n| \cdot |m|} =: n\#m$$

where $|m|$ is the number of bits in the binary representation of m. Bounded arithmetic can be formulated now as $I\Delta_0$ in the language $\mathcal{L}_{BA}$ of bounded arithmetic, that is $\mathcal{L}_{PA}$ extended by $|.|, \#$, or, equivalently, as $I\Delta_0 + \Omega_1$ (where $\Omega_1 \equiv \forall x \exists y(|x|^2 = |y|)$), the latter being the original formulation of bounded arithmetic, see [21]. The provable total functions of bounded arithmetic are the functions computable by a TURING machine in polynomial time using oracles from $\Delta_0(\mathcal{L}_{BA})$, i.e. the polynomial hierarchy.

A stratification of bounded arithmetic, which corresponds to the stratification of the polynomial hierarchy, is obtained by putting restrictions on induction axioms; namely, allowing induction only for certain classes, Σ_i^b, of bounded formulas, and using length induction ($LIND$) in place of successor induction (IND). The most important sub-theories of bounded arithmetic are the theories S_2^i, axiomatized by $\Sigma_i^b - LIND$, and the theories T_2^i, axiomatized by $\Sigma_i^b - IND$. The following is known for these theories:

$$S_2^1 \subseteq T_2^1 \preceq_{\forall \Sigma_2^b} S_2^2 \subseteq T_2^2 \preceq_{\forall \Sigma_3^b} S_2^3 \ldots$$

and their union is the theory $S_2 = T_2 = I\Delta_0(\mathcal{L}_{BA})$ [6, 9]. Here $T \preceq_{\forall \Sigma_i^b} T'$ means that T' is a $\forall \Sigma_i^b$-conservative extension of T. Furthermore, the class of predicates definable by Σ_i^b (or Π_i^b) formulas is precisely the class of predicates in the ith level Σ_i^p (or Π_i^p, resp.) of the polynomial hierarchy. In addition, the Σ_i^b-definable functions of S_2^i are precisely the $\square_i^p$-functions, which are the functions computable in polynomial time using an oracle for Σ_{i-1}^p (cf. [6]).

The main open problem for bounded arithmetic is the question if S_2 is finitely axiomatizable. As S_2^i and T_2^i are finitely axiomatizable, this question is equivalent to ask if there exists an i with $T_2^i = S_2^{i+1}$. This question is also connected to the open problem whether the polynomial hierarchy collapses, hence also to $P =? NP$. The precise connection is that S_2 is finitely axiomatizable if and only if S_2 can prove that the polynomial hierarchy collapses [10, 23]. The common conjecture is that the answer to all these questions is NO!

Restricted consistency notions

We assume familiarity with [6]. From now on let $\mathcal{L}$ be $\mathcal{L}_{BA}$, the first order language of bounded arithmetic. For convenience we assume that $\mathcal{L}$ contains some more symbols for polytime functions (finitely many), e.g. for coding and decoding sequences (e.g. we could take the language L_2 from [17]).

Several restricted consistency notions are known from the literature. Above we have described some of them. The notion of restricted proof studied here will be similar to the notions "i-regular proof" in [12, Definition 10.5.2] and [14, Definition 1.4], and "strictly i-normal proof" in [20, p.81], but combined with a new idea. To explain this let us first explain why usual approaches for proving consistency do not work in weak arithmetic. The reason for this is that in case of the usual feasible coding of syntax (cf. [6]) it is impossible to feasibly evaluate closed terms from the language of bounded arithmetic – their values grow exponentially in their GÖDEL-numbers. What happens if we play with the growth rate of GÖDEL numbering? On the one hand, as mentioned above the usual "feasible coding" of syntax yields $S_2 \nvdash BDCon_{S^1_2}$. On the other hand, if we take a "very unfeasible" sequence coding, e.g. one based on exponentiation like $\langle n_1, \dots, n_k \rangle = 2^{n_1+1} \cdot 3^{n_2+1} \cdot \ldots \cdot p_k^{n_k+1}$, p_k being the k-th prime number, then soundness of S_2-proofs can be proven in weak fragments of bounded arithmetic. We have $S^1_2 \vdash \widetilde{BDCon}_{S_2}$ where in $\widetilde{BDCon}_{S_2}$ syntax is coded in the "very unfeasible" way. Of course, in this setting we loose something, namely based on the very unfeasible coding GÖDEL's incompleteness theorems are not provable, because substitution of terms grows exponentially.

What we will do in this paper is that we will adjust the growth rate in a certain way which allows us to feasibly evaluate GÖDEL numbers of terms, with the cost that GÖDEL's incompleteness theorems will not be provable. But still there will be available enough other properties of formalized provability $\mathrm{RProv}^*_{T^i_2}$ (Definition 9). We will have that S^1_2 proves

$$\mathrm{RProv}^*_{T^i_2}(\ulcorner \Delta, \varphi \urcorner) \text{ and } \mathrm{RProv}^*_{T^i_2}(\ulcorner \Delta, \neg\varphi \urcorner) \text{ implies } \mathrm{RProv}^*_{T^i_2}(\ulcorner \Delta \urcorner)$$

(Lemma 10); that T^i_2-proofs can be normalized

$$\mathrm{RProv}^*_{T^i_2}(\ulcorner \varphi \urcorner) \text{ if and only if } T^i_2 \vdash \varphi$$

(Theorem 11); and that a certain $\exists\Sigma^b_1$-Completeness for S^1_2 holds (Theorem 15), which will be a refinement of

$$S^1_2 \vdash \varphi(\vec{u}) \to \mathrm{RProv}^*_{T^0_2}(\varphi(\vec{I_u}))$$

for $\exists\Sigma^b_1$-formulas φ.

Preservation theorems

The main application of our restricted provability notion in this paper will be that we will prove certain preservation theorems. Other applications may lie in the construction of models of bounded arithmetic with certain properties. In order to explain the preservation theorems we are heading for, let us first fix some notions. For $\mathfrak{L}$-structures M, N we write $M \prec^b_0 N$ iff N is a Δ^b_0-elementary extension of M. We write $M \prec^b_1 N$ iff N is an $\exists s\Sigma^b_1$-elementary extension of M. Here $\exists s\Sigma^b_1$ is the set of all formulas $(\exists x)(\exists y \le t)\varphi$ with $\varphi \in \Delta^b_0$. By $s\Sigma^b_i$ we always mean the prenex (or strict) version of Σ^b_i, etc.

For the following definitions let M, N be $\mathfrak{L}$-structures and T, T' be $\mathfrak{L}$-theories. Let $\log M$ be $\{a \in M : (\exists b \in M)(a \le |b|)\}$. Let $|t|_3$ be $|||t|||$ and let $\log^3 M := \{a \in M : (\exists b \in M)(a \le |b|_3)\}$. Let us call an extension $M \subseteq N$ *log^3-proper*, if $\log^3 M \ne \log^3 N$.

The notion "weak end extension" to be defined next is the natural adaption of the well known notion "end extension" to the setting of bounded arithmetic, cf. [7].

Definition 2. *(Weak end extension) N is called a* weak end extension *of M ($M \subseteq^w_e N$), if N is an extension of M and $\log N$ is an end extension of $\log M$, i.e. for all $a \in \log M$, $b \in \log N$ with $N \vDash b \le a$ we have $b \in \log M$.*

Weak end extensions for models of bounded arithmetic are in some aspects similar to end extensions for (general) models of arithmetic. For example, weak end extensions are always Δ_0^b-elementary.

Let the function ω_1 be defined by $\omega_1(x) = 2^{|x|^2}$, i.e. $\omega_1(x) = x\#x$, and $\omega_1^{(y)}$ be the y-fold iteration of ω_1. We want to define sentences Ω_1^{nst} and Ω_1^{∞} such that Ω_1^{nst} expresses that a nonstandard iteration of ω_1 (or, equivalently, of the smash function $\#$) exists. Ω_1^{∞} should express that an upper bound to all finite iterations of ω_1 exists, but not necessarily a nonstandard iteration. Now $\forall x \exists y(||x|| \cdot c = ||y||)$ expresses that $\forall x\, \omega_1^{(|c|)}(x)$ exists. Hence we let the $\mathcal{L}_{\omega_1\omega}$-sentences $\Omega_1^{\infty}, \Omega_1^{\mathrm{nst}}$ be defined by

$$\Omega_1^{\infty} := \forall x \exists y \bigwedge_{k\in\omega} (||x|| \cdot \underline{k} < ||y||)$$

$$\Omega_1^{\mathrm{nst}} := \exists c \big[\big(\bigwedge_{k\in\omega} (\underline{k} < c) \big) \wedge \forall x \exists y (||x|| \cdot c = ||y||) \big]$$

where $\underline{k}$ is some canonical numeral associated with k.

In the next definition we adopt the notion "1-closeness" (cf. [2]) to the setting of bounded arithmetic. 1-closed models of Peano arithmetic PA satisfy $\neg Con(PA)$ [2]. We will show something similar for 1^b-closed models of bounded arithmetic and theories T_2^i, namely that a model M of $S_2^1 + \Omega_1^{\mathrm{nst}}$ which is 1^b-closed w.r.t. T_2^i cannot be a model of our restricted consistency notion for T_2^i.

Definition 3. *(1^b-Closeness)*
M is called 1^b-closed *w.r.t. T, if for any model N of T such that $M \prec_0^b N$ we have $M \prec_1^b N$.*

With $BL\Sigma_1^b$ we denote the following bounded collection schema:

$$(\forall x \leq |t|)(\exists y)\varphi(x,y) \rightarrow (\exists z)(\forall x \leq |t|)(\exists y \leq z)\varphi(x,y)$$

for $\varphi \in \Sigma_1^b$ which may contain parameters. Buss has shown in [8] that S_2^i and $S_2^i + BL\Sigma_1^b$ have the same $\forall\Pi_1^b$-consequences.

We are now ready to state the preservation theorems which we will prove.

Theorem 4. *Let $1 \leq i \leq j$. The following are equivalent:*

1. *S_2^i is $\forall\Pi_1^b$-separated from T_2^j.*
2. *There is a model of S_2^i which does not have a log^3-proper Δ_0^b-elementary extension to a model of T_2^j.*
3. *There is a model of $S_2^i + \Omega_1^{nst}$ which is 1^b-closed w.r.t. T_2^j.*
4. *There is a countable model of $S_2^i + BL\Sigma_1^b$ without weak end extensions to models of T_2^j.*

The equivalence between 1. and 2. is well-known, it is included just to complete the list. One big impact of Theorem 4 is that solving the main open problem of bounded arithmetic with consistency statements is reduced to constructing models of bounded arithmetic with certain properties.

In the next section we will prove the easy directions of Theorem 4, i.e. the one from 1. to 2., 3., respectively 4. In section 2 we will define a restricted consistency notion of T_2^j – $\mathrm{RCon}^*(T_2^j)$ – which will be the strongest consistency statement (over S_2^1) provable in T_2^j. That is, every $\forall\Pi_1^b$-consequence of T_2^j follows in S_2^1 from $\mathrm{RCon}^*(T_2^j)$. Such consistency notions are known from the literature, e.g. $Con(G_i)$ in [13] or $RCon(T_1^i)$ in [14], or see [12] for a treatment of both. The advantage of our approach over the others is that our consistency notion is for theories in the same language $\mathcal{L}_{BA}$, hence we can use $\mathrm{RCon}^*(T_2^j)$ to (directly) construct models of T_2^j. This is needed in the following 3 sections to complete the proof of Theorem 4. In section 6 we extend our results by adding a $\forall(\Sigma_i^b \cup \Pi_i^b)$-sentence to the theories. In the last section we sketch how the separation problem of bounded arithmetic can be connected to models without *proper* weak end extensions. This is stressed because there is a similar (open) question for models of $I\Delta_0$[2]: Are there models of $I\Delta_0 + B\Sigma_1$ without proper end extensions to models of $I\Delta_0$?[3]

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1 Basic Properties

From the definitions we directly obtain

Lemma 5. *Models which do not have a Δ_0^b-elementary extension to a model of T are 1^b-closed w.r.t. T and do not have weak end extensions to models of T, as weak end extension are Δ_0^b-elementary.* □

Given an $\mathcal{L}$-structure M there is always an elementary extension to an $\mathcal{L}$-structure N such that $N \vDash \Omega_1^{\mathrm{nst}}$. This can be seen by a simple compactness argument applied to

$$eldiag(M) \cup \{\underline{n} < c \,:\, n \in \omega\} \cup \{\forall x \exists y (||x|| \cdot c = ||y||)\}$$

where $eldiag(M)$ denotes the elementary diagram of M. Hence T, $T + \Omega_1^\infty$, $T + \Omega_1^{\mathrm{nst}}$ all prove the same first order sentences.

A similar argumentation as in [2] Remark 1.2 using these observations yields

Lemma 6. *Let $T \subseteq T'$ be $\mathcal{L}$-theories, where $T \subseteq \forall\exists s\Sigma_1^b$, then there exists a countable model of $T + \Omega_1^\infty$ which is 1^b-closed w.r.t. T'.*

Proof. We repeat the argument from [2] Remark 1.2 in an adapted form. Let $t_0, t_1, \dots$ be an enumeration of all triples

$$\langle m, \langle k_1, \dots, k_n \rangle, \varphi \rangle$$

where $m, n, k_1, \dots, k_n \in \mathbb{N}$, $\varphi \in \exists s\Sigma_1^b$ and the number of free variables in φ is n. W.l.o.g. $t_l = \langle m, \langle k_1, \dots, k_n \rangle, \varphi \rangle$ implies $m \leq l$.

We define a tower $M_0 \prec_0^b M_1 \prec_0^b \dots$ of countable models of $T + \Omega_1^\infty$. Let M_0 be a countable model of $T + \Omega_1^{\mathrm{nst}}$. Assume that we have defined $M_0, \dots, M_l$ and enumerations $\{x_k^m : k \in \mathbb{N}\}$ of M_m for $m \leq l$. Let $t_l = \langle m, \langle k_1, \dots, k_n \rangle, \varphi \rangle$.

[2]See [1] for a summary of the most important partial results on the end extension problem.
[3]To the authors best knowledge this question was raised by Kirby and Paris in 1977.

If there is a Δ_0^b-elementary extension M' of M_l which is a model of $T' + \varphi(x^m_{k_1}, \dots, x^m_{k_n})$, then we can find an elementary extension M'' of M' satisfying Ω_1^{nst} (we can restrict ourselves to countable M', M''). Let $M_{l+1} = M''$. Otherwise let $M_{l+1} = M_l$. We fix any enumeration $\{x^{l+1}_k : k \in \mathbb{N}\}$ of M_{l+1}. Now let $M = \bigcup\{M_l : l \in \mathbb{N}\}$. Then $M \models T + \Omega_1^\infty$ which is 1^b-closed w.r.t. T'. □

This construction does *not* produce a model of $T + \Omega_1^{\mathrm{nst}}$ which is 1^b-closed w.r.t. T', because Ω_1^{nst} is "Σ_3", where Ω_1^∞ is only "Π_2", and the model is constructed as a union of a chain of models, which, in general, does not preserve Σ_3-sentences.

One direction of our disired results is easy, that the $\forall\Pi_1^b$-separation yields certain models. The argumentation follows the one from Fact 1.

Theorem 7. *Let $S_2^1 \subseteq T \subseteq T'$, T, T' $\mathcal{L}$-theories. If T is $\forall\Pi_1^b$-separated from T', then there exists a countable model M of $T + \Omega_1^{nst}$ which does not have a Δ_0^b-elementary extension to a model of T'. Hence M is also 1^b-closed w.r.t. T', and also M does not have weak end extensions to models of T'.*

Proof. Under the assumption there is a $\varphi \in \forall s\Pi_1^b$ such that $T' \vdash \varphi$ and $T \nvdash \varphi$ as S_2^1 knows $\forall\Pi_1^b = \forall s\Pi_1^b$. The above remark shows $T + \Omega_1^{\mathrm{nst}} \nvdash \varphi$, hence there is a countable model M of $T + \Omega_1^{\mathrm{nst}} + \neg\varphi$. Now M does not have a Δ_0^b-elementary extension to a model of T', because if $M \prec_0^b M'$, then $M' \models \neg\varphi$ using upwards persistency of $\exists s\Sigma_1^b$-formulas, and therefore $M' \not\models T'$ as $T' \vdash \varphi$.

By Lemma 5 every model which does not have a Δ_0^b-elementary extension to a model of T' is already 1^b-closed w.r.t. T', and also does not have weak end extensions to models of T'. □

This proves directions 1. ⇒ 2. resp. 1. ⇒ 3. of Theorem 4. For 1. ⇒ 4. observe that for $\varphi \in \forall s\Pi_1^b$ with $S_2^i \nvdash \varphi$ and $T_2^j \vdash \varphi$ we also have $S_2^i + BL\Sigma_1^b \nvdash \varphi$ as $S_2^i + BL\Sigma_1^b$ is $\forall\Pi_1^b$-conservative over S_2^i, hence there is a countable model M of $S_2^i + BL\Sigma_1^b + \neg\varphi$. The same argument as in the proof of Theorem 7 shows that there are no weak end extensions of M to models of T_2^j.

We are going to prove converses of Theorem 7. That is, from the existence of certain models we will derive the $\forall\Pi_1^b$-separation of bounded arithmetic theories. The separating sentence will always be a restricted consistency notion of the "stronger" theory which will be defined in the next section. .

2 Restricted proofs

Our notion of restricted proof will be similar to the notions "i-regular proof" in [12, Definition 10.5.2] and [14, Definition 1.4], and "strictly i-normal proof" in [20, p.81]. The main difference between these notions will be that we will vary the coding of formal terms in order to make the coding more "unfeasible" so that values of codes of closed terms are bounded by the codes themselves, where e.g. in [20] restrictions of the provability notion to proofs of small and very small sizes are considered.

We assume that we have constants c_n for each $n \in \omega$ in our language $\mathcal{L}$, and a suitable axiomatization of them in our theories, for example $S_i c_n = c_{S_i n}$ for $i = 0, 1$. Hence in this setting I_u from [6] can simply be defined by $I_u := c_u$.

Let $\ulcorner . \urcorner$ be a usual feasible GÖDELisation as in [6], then $\mathrm{sub}(w, x^\bullet, n)$ – the result of replacing the variable[4] $x^\bullet$ in the string w by the numeral I_n – is a Σ_1^b-definable function in S_2^1 and S_2^1 can prove all necessary properties. We further have $\mathrm{l}(\mathrm{sub}(w, x^\bullet, n)) = \mathrm{l}(w)$, where $\mathrm{l}(w)$ denotes the length of the sequence w, i.e. the number of elements forming w. For formal terms t let $\mathrm{l}(t)$ denote its formal length, i.e. the number of symbols forming t. Obviously we have $\mathrm{l}(t) \leq \mathrm{l}(\ulcorner t \urcorner)$.

One problem with feasible GÖDELisation is that the values of terms have a bigger growth rate that their GÖDEL numbers, e.g.

$$c := \ulcorner \underbrace{2\# \ldots \# 2}_{|n| \text{ times}} \urcorner = 2^{O(|n|)} = n^{O(1)} \qquad \text{but} \qquad val(c) = \Omega(2^n)$$

Of course there cannot be a GÖDELisation G of closed $\mathcal{L}$-terms such that both

1) $\mathrm{val}(c) \leq c^{O(1)}$ for $c \in G$, and

2) $\mathrm{sub}(w, x^\bullet, n) \leq (w + n)^{O(1)}$,

because then $f(w, n) := \mathrm{val}(\mathrm{sub}(w, x^\bullet, n))$ would be a universal bounding function for all polynomially growth rate functions, which itself grows polynomially. By diagonalisation this cannot be possible.

Let us review the growth rates which are achieved via feasible GÖDELisation. Assume that we have a function symbol ω_1 in $\mathcal{L}$ with $\omega_1(n) = n\#n = 2^{|n|^2}$. With $\omega_1^{(k)}(n)$ we denote the k-fold iteration of ω_1. Then $\omega_1^{(|m|)}(n) = 2^{|n|^m} \approx 2_2(||n|| \cdot m)$ with $2_2(n) = 2^{2^n}$. Hence a natural candidate for a GÖDELisation which has property 1) but fails for 2) is

$$\ulcorner t \urcorner^* := 2_2(||\ulcorner t \urcorner|| \cdot 2^{\mathrm{l}(\ulcorner t \urcorner)}) + \ulcorner t \urcorner.$$

(In this definition we can think of $\mathrm{l}(\ulcorner t \urcorner)$ as the number of symbols $\mathrm{l}(t)$ in the formal term t – we have that $\mathrm{l}(\ulcorner t \urcorner)$ is an upper bound to $\mathrm{l}(t)$.) Let $\mathrm{rem}(n)$ be n without its leading bit, then $\ulcorner t \urcorner = \mathrm{rem}(\ulcorner t \urcorner^*)$. We define $\mathrm{val}^*(c)$ and $\mathrm{sub}^*(w, x^\bullet, n)$ to be the value of the closed term c resp. the result of substituting for $x^\bullet$ in w the numeral I_n, this time with respect to $\ulcorner . \urcorner^*$. I.e., with $w' = \mathrm{rem}(w)$ we can write

$$\begin{aligned} \mathrm{val}^*(c) &:= \mathrm{val}(\mathrm{rem}(c)) \\ \mathrm{sub}^*(w, x^\bullet, n) &:= 2_2(||\,\mathrm{sub}(w', x^\bullet, n)|| \cdot 2^{\mathrm{l}(w')}) + \mathrm{sub}(w', x^\bullet, n) \\ \mathrm{l}^*(w) &:= \mathrm{l}(\mathrm{rem}(w)). \end{aligned}$$

As argued above sub^* cannot be polynomially bounded as sub^* and val^* together diagonalise $2^{|n|^k}$. But we observe that for a fixed w with $\mathrm{l}^*(w)$ standard the function $n \mapsto \mathrm{sub}^*(w, x^\bullet, n)$ is polytime because $\mathrm{sub}^*(w, x^\bullet, n)$ can be bounded by $\omega_1^{(\mathrm{l}^*(w)+O(1))}(n)$.

Now we can prove property 1) for $\ulcorner . \urcorner^*$

Lemma 8. $\mathrm{val}^*(c) \leq c$.

[4] We use $x^\bullet$ to speak about codes indicating variables.

Proof. We identify function symbols with the functions they represent. Then the assertion follows from

$$|t| < |\ulcorner t \urcorner^*|$$

for closed $\mathfrak{L}$-terms t. All function symbols f in $\mathfrak{L}$ represent polytime functions, hence there are constants c_f such that

$$|f(\vec{n})| < \max(|\vec{n}|, 2)^{c_f}.$$

Assume $t = fu_1 \dots u_k$, then $||\ulcorner t \urcorner|| \geq \max(||\ulcorner \vec{u} \urcorner||)$ and, w.l.o.g., $\mathrm{l}(\ulcorner t \urcorner) \geq \max(\mathrm{l}(\ulcorner \vec{u} \urcorner)) + |c_f|$, hence we obtain inductively

$$\begin{aligned} |t| &< \max(|\vec{u}|, 2)^{c_f} \leq 2^{\max(||\vec{u}||,1) \cdot c_f} \\ &\overset{i.h.}{\leq} 2^{\max_{i=1,\dots,k}(||\ulcorner u_i \urcorner|| \cdot 2^{\mathrm{l}(\ulcorner u_i \urcorner)}) \cdot c_f} \\ &\leq 2^{\max(||\ulcorner \vec{u} \urcorner||) \cdot \max(2^{\mathrm{l}(\ulcorner \vec{u} \urcorner)}) \cdot c_f} \\ &\leq 2^{||\ulcorner t \urcorner|| \cdot 2^{\mathrm{l}(\ulcorner t \urcorner)}} = |\ulcorner t \urcorner^*|. \end{aligned}$$

□

The form of restricted proofs we will consider here is similar to the notions "i-regular proof" in [12, Definition 10.5.2] and [14, Definition 1.4], and "strictly i-normal proof" in [20, p.81].

Definition 9. $\mathrm{RProof}^*_{T^i_2}(\langle p, \vec{b}, \vec{t}, \vec{d}, \vec{T}, \vec{D} \rangle, \Delta)$ *holds if the following conditions are satisfied:*

1. *$p, \vec{b}, \Delta$ are coded via $\ulcorner . \urcorner$; $\vec{t}, \vec{d}, \vec{T}, \vec{D}$ via $\ulcorner . \urcorner^*$.*
2. *p is a* Tait*-style*[5] *derivation of the set Δ using the $s\Sigma^b_i - IND$-rule, and all formulas in p are in Σ^b_∞.*
3. *All cut formulas in p are in $s\Sigma^b_i \cup s\Pi^b_i$.*
4. *p is in free variable normal form.*
5. *If $\vec{a}$ are all parameters (i.e. free variables in Δ) and $\vec{b} = (b_0, \dots, b_{k-1})$ all other variables in p, then*
   (a) *if the elimination inference of b_i is below the elimination inference of b_j, then $i < j$.*
   (b) *$\vec{t}$ is a k-tuple of monotone terms with variables among $\vec{a}$; $\vec{d}$ is a k-tuple of proofs.*
   (c) *the elimination inference of b_i is one of*

$$(s\Sigma^b_i - IND) \qquad \frac{\Delta, \neg A(b_i), A(b_i + 1)}{\Delta, \neg A(0), A(r(\vec{a}, b_0, \dots, b_{i-1}))}$$

$$(\forall \leq) \qquad \frac{\Delta, b_i \not\leq r(\vec{a}, b_0, \dots, b_{i-1}), A(b_i)}{\Delta, \forall x \leq r(\vec{a}, b_0, \dots, b_{i-1}) A(x)}$$

[5]With "Tait-style" we mean that sets (or formally: sequences) of formulas are derived, and that negation is a syntactic operation, not a symbol of our formal language.

and d_i is a proof of

$$b_0 \not\leq t_0(\vec{a}), \dots, b_{i-1} \not\leq t_{i-1}(\vec{a}), r(\vec{a}, b_0, \dots, b_{i-1}) \leq t_i(\vec{a})$$

that is without the IND-rule, is quantifier-free and contains only the variables $b_0, \dots, b_{i-1}$. (E.g., $t_i = \sigma[r]_{b_0,\dots,b_{i-1}}(t_0, \dots, t_{i-1})$ with σ the metafunction defined in [6].)

(d) *We define the set $Bd_\varphi^{\vec{b},\vec{t}}(T)$ for bounded φ via sets A_φ and B_φ, which are defined by recursion on φ:*

φ	A_φ	B_φ
$P\vec{u}$ *atomic*	$\emptyset$	$\{\vec{u}\}$
$\psi \circ \chi$	$A_\psi \cup A_\chi$	$B_\psi \cup B_\chi$
$Qx \leq u\psi(x)$	$\big(A_{\psi(x)}\big)_x(a_\varphi) \cup \{a_\varphi \not\leq u\}$	$\big(B_{\psi(x)}\big)_x(a_\varphi) \cup \{u\}$

Then

$$Bd_\varphi^{\vec{b},\vec{t}}(T) := \{b_i \not\leq t_i \;:\; i \leq k\} \cup A_\varphi \cup \{\bigwedge_{u \in B_\varphi} (u \leq T)\}$$

expresses, that T is an upper bound to the "world" of φ. By this we mean that all values which are considered by φ are bounded by T.
$\vec{T}$ is a list of monotone terms with variables among $\vec{a}$, $\vec{D}$ is a list of proofs similar to 5c), such that for every ψ in p there is some T_ψ in $\vec{T}$ and a proof d_ψ of $Bd_\varphi^{\vec{b},\vec{t}}(T_\psi)$ in $\vec{D}$, which contains only the same free variables as $Bd_\varphi^{\vec{b},\vec{t}}(T_\psi)$.
This means that $\max(\vec{T})$ bounds the "world" of p, which allows us to compute all values occuring in a soundness proof of a $\mathrm{RProof}^*_{T_2^i}$-derivation, if we are able to compute $\mathrm{val}(\max(\vec{T}))$.

Now we define

$$\mathrm{RProv}^*_{T_2^i}(\Delta) := \exists P\, \mathrm{RProof}^*_{T_2^i}(P, \Delta)$$
$$\mathrm{RCon}^*(T_2^i) := \neg\, \mathrm{RProv}^*_{T_2^i}(\ulcorner \emptyset \urcorner).$$

If $\mathrm{RProof}^*_{T_2^i}(P, \ulcorner \Delta \urcorner)$ then we say that P *is a restricted-T_2^i-proof of* Δ.

Lemma 10. (S_2^1) *If $\varphi \in s\Sigma_i^b$, $\mathrm{RProv}^*_{T_2^i}(\ulcorner \Delta, \varphi \urcorner)$ and $\mathrm{RProv}^*_{T_2^i}(\ulcorner \Delta, \neg\varphi \urcorner)$, then $\mathrm{RProv}^*_{T_2^i}(\ulcorner \Delta \urcorner)$.* □

Theorem 11. *(Normalizing)*
$\mathrm{RProv}^*_{T_2^i}(\ulcorner \varphi \urcorner) \Leftrightarrow T_2^i \vdash \varphi$.

Proof. $\Rightarrow$: is clear.
$\Leftarrow$: We give two proofs. We will sketch a prooftheoretic one, and give a precise modeltheoretic one because we will extend the model-theoretic method later on.

Prooftheoretic proof: Assume $T_2^i \vdash \varphi$. Then there is a derivation d_1 of φ which has as axioms instances of $BASIC$ and which uses the $s\Sigma_i^b$-IND-rule. By partial cutelimination and further normalisations (see [6] or [3, 5]) we obtain a derivation d_2 of φ, in which cutformulas are in $s\Sigma_i^b \cup s\Pi_i^b$ and the only free variables in d_2 are those occuring free in φ (the parameters) and the eigenvariables of d_2. By renaming eigenvariables and collecting data we obtain a restricted-T_2^i-proof of φ.

Modeltheoretic proof: Assume $\neg\,\mathrm{RProv}^*_{T_2^i}(\ulcorner\varphi\urcorner)$. Via modeltheoretic forcing (cf. [11]) we construct a countable model N of $T_2^i + \neg\varphi$. Hence $T_2^i \nvdash \varphi$.

We start adding new constants $(d_n)_{n\in\omega}$ (the witnesses of our forcing construction) to our language (call it $\mathcal{L}^+$). Again we call the formalized restricted proof predicates in this extended language $\mathrm{RProof}^*_{T_2^i}$ and $\mathrm{RProv}^*_{T_2^i}$. Then again $\neg\,\mathrm{RProv}^*_{T_2^i}(\ulcorner\varphi\urcorner)$, because the new constants can be replaced by 0.

We shall define an increasing chain $(T_n)_{n\in\omega}$ of finite sets of $\mathcal{L}^+$-sentences, such that for every n

$$\neg\,\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_n\urcorner). \tag{1}$$

In the end their union $T^+ := \bigcup_n T_n$ will be a HINTIKKA set for $\mathcal{L}^+$ (cf. section 2.3 in [11]), and the canonical model of the atomic sentences in T^+ will be a model of $T_2^i + \neg\varphi$. To ensure that T^+ will have these properties, we carry out several tasks as we build the chain. We consider the following tasks:

$(T1)$ $\neg\varphi \in T^+$

$(T2)$ For all $s\Sigma_i^b(\mathcal{L}^+)$-sentences ψ, $\psi \in T^+$ or $\neg\psi \in T^+$.

$(T3)_{\psi(x),t}$ (for ψ a $s\Sigma_i^b(\mathcal{L}^+)$-formula in one free variable and $t \in \mathcal{L}^+$ closed.)
$\psi(0), \neg\psi(t) \in T^+ \Rightarrow \psi(s), \neg\psi(s+1) \in T^+$ for some closed $s \in \mathcal{L}^+$.

The following tasks (together with $(T2)$ will insure that T^+ is a HINTIKKA set:

$(H1)_{\psi,\chi}$ $\psi \wedge \chi \in T^+ \Rightarrow \psi, \chi \in T^+$

$(H2)_{\psi,\chi}$ $\psi \vee \chi \in T^+ \Rightarrow \psi \in T^+$ or $\chi \in T^+$

$(H3)_{\forall x\psi(x)}$ $\forall x\psi(x) \in T^+ \Rightarrow \psi(t) \in T^+$ for every closed $t \in \mathcal{L}^+$.

$(H4)_{\exists x\psi(x)}$ $\exists x\psi(x) \in T^+ \Rightarrow \psi(t) \in T^+$ for some closed $t \in \mathcal{L}^+$.

If these tasks are all carried out, then T^+ is a HINTIKKA set. E.g. if $\psi \in s\Sigma_i^b$ and $\psi \in T^+$, then $\neg\psi \notin T^+$, because otherwise $\psi, \neg\psi \in T_n$ for some n and obviously $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg\psi,\psi\urcorner)$ contradicting condition (1) for n. Or, if $s = t \in T^+$ for some closed terms $s, t \in \mathcal{L}^+$ then also $t = s \in T^+$ using $(T2)$ and $\mathrm{RProv}^*_{T_2^i}(\ulcorner s \neq t, t = s\urcorner)$.

Write N^+ for the canonical model of the atomic sentences in T^+. Then N^+ is a model of T^+. Using $(T2)$ N^+ is a model of the $s\Sigma_i^b \cup s\Pi_i^b$-consequences of T_2^i. Also N^+ fulfills $s\Sigma_i^b$-IND, because if $\psi \in s\Sigma_i^b$ and $N^+ \vDash \psi(0) \wedge \exists x\neg\psi(x)$, then there is some closed term t such that $N^+ \vDash \psi(0) \wedge \neg\psi(t)$ as N^+ is a term model. With $(T2)$ we get $\psi(0), \neg\psi(t) \in T^+$. By $(T3)_{\psi(x),t}$ there is a closed $\mathcal{L}^+$-term s such that $\psi(s), \neg\psi(s+1) \in T^+$. Hence $N^+ \vDash \exists x(\psi(x) \wedge \neg\psi(x+1))$. Altogether this shows

$$N^+ \vDash \psi(0) \wedge \exists x\neg\psi(x) \to \exists x(\psi(x) \wedge \neg\psi(x+1)),$$

thus

$$N^+ \vDash \psi(0) \wedge \forall x(\psi(x) \to \psi(x+1)) \to \forall x\psi(x).$$

By $(T1)$ we further get $N^+ \models \neg\varphi$. Hence N^+ is the desired model.

It remains to show that all the countable tasks are enforceable. This means that for any task $\mathcal{T}$ and any infinite set $X \subset \omega$ such that $\omega \setminus X$ is also infinite and $0 \notin X$, we consider the following game $G(\mathcal{T}, X)$. The player, $\forall$ and $\exists$, pick the sets T_n in turn; player $\exists$ makes the choice of T_n if and only if $n \in X$. Player $\exists$ wins if T^+ has property $\mathcal{T}$, otherwise $\forall$ wins. Now "enforceable" means that in all these games player $\exists$ has a winning strategy.

Consider a task. We describe an expert (feminine by convention) handling this task. Let $X \subset \omega$ be her subset and assume $n \in X$.

$(T1)$ Let $T_0 := \{\neg\varphi\}$ (*one* expert is allowed to have 0 in her set X).

$(T2)$ Let her list as $(\psi_n : n \in X)$ all sentences of $s\Sigma_i^b(\mathcal{L}^+)$.
If $\neg\,\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_{n-1}, \neg\psi_n\urcorner)$, let $T_n := T_{n-1} \cup \{\psi_n\}$. Otherwise she sets $T_n := T_{n-1} \cup \{\neg\psi_n\}$. Thus she fulfills her task.

We have to check that condition (1) is not violated. Suppose for the sake of contradiction that we have $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_n\urcorner)$, then by construction $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_{n-1}, \neg\psi_n\urcorner)$ and $T_n = T_{n-1} \cup \{\neg\psi_n\}$. Now we obtain $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_{n-1}\urcorner)$ using Lemma 10 contradicting condition (1) for $n-1$.

$(T3)_{\psi(x),t}$ If $\psi(0), \neg\psi(t) \in T_{n-1}$, then she chooses some witness d not occurring in T_{n-1} and sets $T_n := T_{n-1} \cup \{\psi(d), \neg\psi(d+1)\}$. Otherwise she does nothing. This strategy works, because her subset X contains arbitrarily large numbers.

Condition (1) is fulfilled, because if we have $\psi(0), \neg\psi(t) \in T_{n-1}$ and $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_n\urcorner)$, then by construction we can replace d in T_n by a fresh variable a obtaining $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_{n-1}, \neg\psi(a), \psi(a+1)\urcorner)$, hence one application of formalized $s\Sigma_i^b$-IND (in $\mathrm{RProv}^*_{T_2^i}$) yields $\mathrm{RProv}^*_{T_2^i}(\ulcorner\neg T_{n-1}\urcorner)$.

Tasks $(H1)$–$(H4)$ are treated in a similar way. □

Theorem 12. $(i \geq 0)$

$$T_2^i \vdash \mathrm{RCon}^*(T_2^i).$$

Proof. The argument is similar to [12] Theorem 10.5.3.

Suppose for the sake of contradiction $P = \langle p, \vec{b}, \vec{t}, \vec{d}, \vec{T}, \vec{D}\rangle$ is a restricted-T_2^i-proof of the empty set, then by the sub formula property all formulas in p are in $s\Sigma_i^b \cup s\Pi_i^b$. Now S_2^1 can evaluate all $\vec{t}, \vec{T}$ (cf. Lemma 8). Consider a partial truth definition $\mathrm{Tr}_i(x,y)$ for $s\Sigma_i^b$-formulas, ($\mathrm{Tr}_i \in s\Sigma_i^b$), then S_2^1 can prove TARSKI's conditions for all formulas in p with $\vec{b} \leq \vec{t}$ using $\vec{T}$ and $\vec{D}$. Extend Tr_i to STr_i for sets of $s\Sigma_i^b \cup s\Pi_i^b$-formulas ($\mathrm{STr}_i \in \Delta_{i+1}^b$), and consider the $s\Pi_{i+1}^b$-formula $\varphi(s)$

$$\forall r \leq s \forall b_0 \leq t_0 \ldots \forall b_k \leq t_k \, \mathrm{STr}_i(S_r, \vec{b}) \tag{2}$$

where $p = (S_0, \ldots, S_l)$. Local correctness, i.e. $\varphi(s) \to \varphi(s+1)$, is provable in T_2^i. With $s\Pi_{i+1}^b - LIND$ we can show $\varphi(l)$ which implies $\mathrm{Tr}_i(\emptyset)$ – a contradiction. Hence S_2^{i+1} proves $\mathrm{RCon}^*(T_2^i)$, and as S_2^{i+1} is $\forall\Sigma_{i+1}^b$-conservative over T_2^i, the same holds for T_2^i. □

Remark 13. *The* GÖDEL *sentences as proved in [6] as well as the following results in this article admit another kind of restricting proof predicates. By adding weights we can force the depth* $\mathrm{dp}(p)$ *of proofs* p *to be bounded by* $||p||$,

$$\mathrm{dp}(p) \leq ||p||.$$

Then the assertion (2) *in the proof of Theorem 12 can be reformulated as*

$$\tilde{\varphi}(s) := \forall r_1, r_2 \leq l \, \forall b_0 \leq t_0 \ldots \forall b_k \leq t_k \big[r_1 \leq r_2 \leq r_1 + 2^s \& \\ \forall h \leq r_1 \forall j \leq l_h \, \mathrm{STr}_i(S_{j,h}, \vec{b}) \to \forall h \leq r_2 \forall j \leq l_h \, \mathrm{STr}_i(S_{j,h}, \vec{b}) \big]$$

if p *is written as* $(S_{0,0}, S_{1,0}, \ldots, S_{l_0,0}, S_{0,1}, \ldots, S_{l_l,l})$ *with* $j =$ *depth of* $S_{i,j}$ *in* p. *Thus* $\Pi^b_{i+1} - L^3IND$ *suffices to prove* $\tilde{\varphi}(|l|)$ *from* $\varphi(0)$, *as* $\tilde{\varphi}(s) \in \Pi^b_{i+1}$ *and* $l \leq ||p||$. *Hence for any theory* $T = \Sigma^b_i - L^k IND$ *we get*

$$T + \Pi^b_{i+1} - L^3IND \vdash \mathrm{RCon}^*(T).$$

□

Remember that we have $n \mapsto \mathrm{sub}^*(\ulcorner\theta\urcorner^*, x^\bullet, n)$ as a band of polytime functions indexed by θ. Define

$$\mathrm{l}^*(\langle p, \vec{b}, \vec{t}, \vec{d}, \vec{T}, \vec{D} \rangle) := \max\big(\mathrm{l}^*(\vec{t}), \mathrm{l}^*(\vec{d}), \mathrm{l}^*(\vec{T}), \mathrm{l}^*(\vec{D})\big)$$

where, e.g., $\mathrm{l}^*(t_1, \ldots, t_k)$ means $\mathrm{l}^*(t_1), \ldots, \mathrm{l}^*(t_k)$. An inspection of [6, Lemma 7.4.5] and [6, Theorem 7.4.4] shows

Lemma 14. *For all terms* t *with free variables among* $x_1, \ldots, x_l$ *there is a constant* $k_t \in \omega$ *and a term* s_t *with free variables among* $x_1, \ldots, x_l$ *such that*

$$S^1_2 \vdash \exists P \leq s_t(\vec{u}) \big[\mathrm{l}^*(P) \leq k_t \wedge \mathrm{RProof}^*_{T^0_2}(P, \ulcorner t(\vec{I_u}) = I_{t(\vec{u})} \urcorner) \big].$$

Proof. Inspecting Lemma 7.4.5 in [6] shows that the lengths of all terms in the generated proof of $t(\vec{I_u}) = I_{t(\vec{u})}$ is bounded by $\mathrm{l}(t) + o(1)$. Here it is important that in our setting we have $I_u := c_u$ and hence $\mathrm{l}(I_u) = 1$ for arbitrary u (even if u is nonstandard).

In the proof p of $t(\vec{I_u}) = I_{t(\vec{u})}$ generated in Lemma 7.4.5 of [6] only equations occur. Thus we may take $\vec{T}$ as the sequence of all $v + w$ for $v = w$ occuring in p, and $\vec{D}$ as the sequence of simple proofs of $v \leq v + w \wedge w \leq v + w$. Then $\langle \ulcorner p \urcorner, \ulcorner \emptyset \urcorner, \ulcorner \emptyset \urcorner^*, \ulcorner \emptyset \urcorner^*, \ulcorner \vec{T} \urcorner^*, \ulcorner \vec{D} \urcorner^* \rangle$ is a restricted proof of $\ulcorner t(\vec{I_u}) = I_{t(\vec{u})} \urcorner$.

Hence we can choose $k_t = \mathrm{l}(\ulcorner t \urcorner) + o(1)$. □

Theorem 15. ($\exists\Sigma^b_1$*-Completeness for* S^1_2 *w.r.t.* $\ulcorner . \urcorner^*$)

1. *For* $\varphi \in \Sigma^b_1$ *with free variables among* $x_1, \ldots, x_l$ *there exists a constant* $k_\varphi \in \omega$ *and a term* s_φ *with free variables among* $x_1, \ldots, x_l$ *such that*

$$S^1_2 \vdash \varphi(\vec{u}) \to \exists P \leq s_\varphi(\vec{u}) \big[\mathrm{l}^*(P) \leq k_\varphi \wedge \mathrm{RProof}^*_{T^0_2}(P, \ulcorner \varphi(\vec{I_u}) \urcorner) \big].$$

2. *For* $\varphi \in \exists\Sigma^b_1$ *with free variables among* $x_1, \ldots, x_l$ *there exists a constant* $k_\varphi \in \omega$ *such that*

$$S^1_2 \vdash \varphi(\vec{u}) \to \exists P \big[\mathrm{l}^*(P) \leq k_\varphi \wedge \mathrm{RProof}^*_{T^0_2}(P, \ulcorner \varphi(\vec{I_u}) \urcorner) \big].$$

Proof. An inspection of Theorem 7.4.4 in [6] shows that the lengths of terms in generated proofs does not exceed those generated in Lemma 14. Hence we can choose $k_\varphi = \mathrm{l}(\ulcorner\varphi\urcorner) + o(1)$.

Furthermore, proofs are generated by induction on the complexity of φ, they do not contain applications of induction rules, and the structure of $(\forall \leq)$-inferences and their bounds reflect the structure of the bounded $\forall$-quantifiers and their bounds in φ. Thus we can define "from the outside" lists $\vec{b}, \vec{t}$ and $\vec{d}$ (all standard objects), such that if $\varphi(\vec{u})$ holds, then

$$\langle \ulcorner p\urcorner, \ulcorner\vec{b}\urcorner, \ulcorner\vec{t}\urcorner^*, \ulcorner\vec{d}\urcorner^*, \ulcorner\vec{T}\urcorner^*, \ulcorner\vec{D}\urcorner^*\rangle$$

(where $p, \vec{T}, \vec{D}$ are collected similar as in the proof of Lemma 14) is a restricted proof of $\ulcorner\varphi(\vec{I_u})\urcorner$. □

As a corollary we obtain that $\mathrm{RCon}^*(T^i_2)$ is the strongest $\forall\Pi^b_1$-statement (over S^1_2) provable in T^i_2, similar to $Con(G_i)$ in [13] or $RCon(T^i_1)$ in [14], or see [12] for a treatment of both.

Theorem 16. $(i \geq 1)$ *For all $\varphi \in \forall\Pi^b_1$ such that $T^i_2 \vdash \varphi$ we have $S^1_2 \vdash \mathrm{RCon}^*(T^i_2) \to \varphi$.*

Proof. Let $S^1_2 \vdash \varphi \leftrightarrow \forall x \psi(x)$ with $\psi \in s\Pi^b_1$, then we have $T^i_2 \vdash \psi(x)$, hence $\mathrm{RProof}^*_{T^i_2}(P, \ulcorner\psi(x^\bullet)\urcorner)$ for some standard P using Normalizing (Theorem 11). S^1_2 can check that $\mathrm{RProof}^*_{T^i_2}(I_P, \ulcorner\psi(x^\bullet)\urcorner)$ holds. Now $\mathrm{l}^*(P)$ is standard, hence we can substitute values for $x^\bullet$ in P, because $u \mapsto \mathrm{sub}^*(P, x^\bullet, u)$ is polytime. Thus we obtain $S^1_2 \vdash \forall u\, \mathrm{RProv}^*_{T^i_2}(\ulcorner\psi(I_u)\urcorner)$. The $\exists\Sigma^b_1$-completeness of S^1_2 shows $S^1_2 \vdash \neg\psi(u) \to \mathrm{RProv}^*_{T^i_2}(\ulcorner\neg\psi(I_u)\urcorner)$. Hence $S^1_2 \vdash \neg\psi(u) \to \neg\,\mathrm{RCon}^*(T^i_2)$. □

Corollary 17. *Let $i \geq 1$.*

1. $S^1_2 + \mathrm{RCon}^*(T^i_2) =_{\forall\Pi^b_1} T^i_2$.
2. $S^i_2 \neq_{\forall\Pi^b_1} T^j_2 \Leftrightarrow S^i_2 \nvdash \mathrm{RCon}^*(T^j_2)$.

3 Δ^b_0-elementary extension

As shown in Fact 1 it is well-known that the $\forall\Pi^b_1$-separation of S^i_2 from T^j_2 $(1 \leq i \leq j)$ is equivalent to the existence of a model of S^i_2 which does not have a Δ^b_0-elementary extension to a model of T^j_2.

We repeat this argument in an adapted form to prove the next theorem. We do this also because the proof of Theorem 18 is the bases of following proofs.

Theorem 18. $(i \geq 1)$ *Assume M is a model of S^1_2 which does not have a log^3-proper Δ^b_0-elementary extension to a model of T^i_2. Then $M \vDash \neg\,\mathrm{RCon}^*(T^i_2)$.*

Proof. Suppose for the sake of contradiction $M \vDash \mathrm{RCon}^*(T^i_2)$. Let $\mathcal{L}_M$ be the extension of the constants $c_n : n \in \omega$ of our language $\mathcal{L}$ to $c_d : d \in M$, plus some new constant b. So $I_d = c_d$ for all $d \in M$. Consider

$$T := T^i_2 + \Delta^b_0\text{-diag}(M) + \{|I_d|_3 < |b|_3 \ : \ d \in M\},$$

where Δ_0^b-diag(M) is the Δ_0^b-diagram of M

$$\Delta_0^b\text{-diag}(M) := \{\theta(I_{a_1}, \dots, I_{a_n}) \;:\; \theta \in \Delta_0^b, a_1, \dots, a_n \in M, M \models \theta(\vec{a})\}.$$

We will show that T is consistent. Suppose for the sake of contradiction T is inconsistent. Then by compactness there is a finite subset $D \subset \Delta_0^b$-diag(M) and some finite $N \subset M$ such that $T_2^i + D + \{|I_d|_3 < |b|_3 \;:\; d \in N\}$ is inconsistent. Let $d_0 = \max(N)$ and $d_1 = d_0 \# d_0 \in M$. As b has been new, we can replace it by I_{d_1}. Now $|d_1|_3 > |d_0|_3 \geq |d|_3$ for any $d \in N$, hence we can assume w.l.o.g. that $T_2^i + D$ is inconsistent. Using that Δ_0^b is closed under conjunction, we even can assume w.l.o.g. that there is one $\theta(I_{a_1}, \dots, I_{a_n}) \in \Delta_0^b$-diag$(M)$ such that $T_2^i + \theta(I_{a_1}, \dots, I_{a_n})$ is inconsistent. Hence

$$T_2^i \vdash \neg\theta(I_{a_1}, \dots, I_{a_n}).$$

Using Normalizing (Theorem 11) there is a standard restricted-T_2^i-proof with parameters from M deriving this formula, hence

$$M \models \mathrm{RProv}^*_{T_2^i}(\ulcorner \neg\theta(I_{a_1}, \dots, I_{a_n})\urcorner).$$

By Σ_1^b-Completeness (Theorem 15) we have $M \models \mathrm{RProv}^*_{T_2^i}(\ulcorner \theta(I_{a_1}, \dots, I_{a_n})\urcorner)$. Hence using a bounded cut (i.e. a refinement of Lemma 10) we obtain $M \models \mathrm{RProv}^*_{T_2^i}(\emptyset)$ contradicting our assumption $M \models \mathrm{RCon}^*(T_2^i)$. Thus T is consistent.

Let N be a model of T, then (up to isomorphism) $M \prec_0^b N$ because T includes the Δ_0^b-diagram of M. Furthermore $N \models T_2^i + \{|I_d|_3 < |b|_3 \;:\; d \in M\}$, hence N is a $\log^3$-proper extension of M and N is a model of T_2^i, contradicting our assumption that M does not have a $\log^3$-proper Δ_0^b-elementary extension to a model of T_2^i.

Thus our assumption was wrong, and we have $M \models \neg\,\mathrm{RCon}^*(T_2^i)$. □

This theorem can be used to reobtain Fact 1 (and, therefore, 2. $\Rightarrow$ 1. of Theorem 4), i.e. that if there is a model of S_2^i which does not have a $\log^3$-proper Δ_0^b-elementary extension to a model of T_2^j then S_2^i is $\forall\Pi_1^b$-separated from T_2^j $(1 \leq i \leq j)$. Because, under the assumption the last Theorem shows that S_2^i does not prove $\mathrm{RCon}^*(T_2^j)$, while T_2^j does prove it (cf. Theorem 12).

4 1^b-closed models

1-closed models of PA satisfy $\neg Con(PA)$ [2]. Here we show something similar for 1^b-closed models of bounded arithmetic and theories T_2^i.

Theorem 19. $(i \geq 1)$ *Assume M is a model of $S_2^1 + \Omega_1^{nst}$ which is 1^b-closed w.r.t. T_2^i. Then $M \models \neg\,\mathrm{RCon}^*(T_2^i)$.*

Proof. The proof is similar to that of Theorem 1.1 in [2].

Suppose for the sake of contradiction $M \models \mathrm{RCon}^*(T_2^i)$. Fix a nonstandard $b \in M$ such that $M \models \forall x \exists y(||x|| \cdot b = ||y||)$. Let $\psi(z, \ulcorner\varphi\urcorner, x)$ be the formula

$$\forall P\big[1^*(P) < |z| \rightarrow \neg\,\mathrm{RProof}^*_{T_2^i}(P, \ulcorner \neg\varphi(I_z, I_x, y^\bullet)\urcorner)\big]$$

where $\varphi \in s\Sigma_1^b$.

We assert that ψ is universal for (standard) $\exists s\Sigma_1^b$-formulas with one free variable and parameter b, i.e.

$$M \vDash \psi(b, \ulcorner\varphi\urcorner, a) \leftrightarrow \exists y \varphi(b, a, y) \tag{3}$$

for $\varphi \in s\Sigma_1^b$ and $a \in M$.

Fix $\varphi \in s\Sigma_1^b$ and $a \in M$.

$\Leftarrow$: Assume there exists some $d \in M$ such that $M \vDash \varphi(b, a, d)$. By Σ_1^b-Completeness (Theorem 15) there exists $P \in M$ with

$$M \vDash \mathrm{RProof}^*_{T_2^i}(P, \ulcorner\varphi(I_b, I_a, I_d)\urcorner). \tag{4}$$

Suppose for the sake of contradiction $M \nvDash \psi(b, \ulcorner\varphi\urcorner, a)$, i.e. there is a $Q \in M$ such that $M \vDash \mathrm{l}^*(Q) < |b|$ and $M \vDash \mathrm{RProof}^*_{T_2^i}(Q, \ulcorner\neg\varphi(I_b, I_a, y^\bullet)\urcorner)$. As $M \vDash \mathrm{l}^*(Q) < |b|$ and $M \vDash \forall x \exists y(||x|| \cdot b = ||y||)$, we can substitute $y^\bullet$ in Q by I_d obtaining some $Q' \in M$ such that

$$M \vDash \mathrm{RProof}^*_{T_2^i}(Q', \ulcorner\neg\varphi(I_b, I_a, I_d)\urcorner).$$

Now a bounded cut with (4) (cf. Lemma 10) yields $M \vDash \mathrm{RProv}^*_{T_2^i}(\emptyset)$ which contradicts our assumption $M \vDash \mathrm{RCon}^*(T_2^i)$.

$\Rightarrow$: Assume $M \vDash \psi(b, \ulcorner\varphi\urcorner, a)$. Let $\mathcal{L}_M$ be the extension of the constants $c_n : n \in \omega$ of our language $\mathcal{L}$ to $c_d : d \in M$. So $I_d = c_d$ for all $d \in M$.

Consider

$$T \quad := \quad T_2^i + \Delta_0^b\text{-diag}(M) + \exists y \varphi(I_b, I_a, y).$$

We will show that T is consistent. Suppose for the sake of contradiction T is inconsistent. By compactness and the closure of Δ_0^b under conjunction there is some $\theta(I_{a_1}, \ldots, I_{a_n}) \in \Delta_0^b\text{-diag}(M)$ such that $T_2^i + \theta(I_{a_1}, \ldots, I_{a_n}) + \exists y \varphi(I_b, I_a, y)$ is inconsistent, hence

$$T_2^i \vdash \neg\theta(I_{a_1}, \ldots, I_{a_n}), \neg\varphi(I_b, I_a, y).$$

Using Normalizing (Theorem 11) there is a standard restricted-T_2^i-proof with parameters from M for this set of formulas, hence there is some $P \in M$ such that $M \vDash \mathrm{l}^*(P) < |b|$ and

$$M \vDash \mathrm{RProof}^*_{T_2^i}(P, \ulcorner\neg\theta_1(I_{a_1}, \ldots, I_{a_n}), \neg\varphi(I_b, I_a, y)\urcorner).$$

By Σ_1^b-Completeness (Theorem 15) there exist $P' \in M$ such that $\mathrm{l}^*(P')$ is standard and

$$M \vDash \mathrm{RProof}^*_{T_2^i}(P', \ulcorner\theta(I_{a_1}, \ldots, I_{a_n})\urcorner).$$

Hence using a bounded cut (i.e. a refinement of Lemma 10) yields a $Q \in M$ such that $M \vDash \mathrm{l}^*(Q) < |b|$ and

$$M \vDash \mathrm{RProof}^*_{T_2^i}(Q, \ulcorner\neg\varphi(I_b, I_a, y)\urcorner)$$

contradicting our assumption $M \vDash \psi(b, \ulcorner\varphi\urcorner, a)$. Thus T is consistent.

Let N be a model of T, then (up to isomorphism) $M \prec_0^b N$ because T includes the Δ_0^b-diagram of M. Furthermore $N \vDash T_2^i + \exists y \varphi(b, a, y)$, hence $M \vDash$

$\exists y \varphi(b, a, y)$ by our assumption that M is 1^b-closed w.r.t. T_2^i. This finishes the proof of (3).

Considering $\neg\psi(z, x, x)$ there is some $\varphi \in s\Sigma_1^b$ such that $S_2^1 \vdash \exists y \varphi(z, x, y) \leftrightarrow \neg\psi(z, x, x)$. By (3) we have

$$M \models \psi(b, \ulcorner\varphi\urcorner, a) \leftrightarrow \exists y \varphi(b, a, y) \leftrightarrow \neg\psi(b, a, a)$$

for all $a \in M$. Instantiating this with $a := \ulcorner\varphi\urcorner$ we get

$$M \models \psi(b, \ulcorner\varphi\urcorner, \ulcorner\varphi\urcorner) \leftrightarrow \neg\psi(b, \ulcorner\varphi\urcorner, \ulcorner\varphi\urcorner),$$

a contradiction.

Thus our assumption was wrong, and we have $M \models \neg\,\mathrm{RCon}^*(T_2^i)$. $\square$

Corollary 20. $(1 \leq i \leq j)$ *Assume that there is a model of $S_2^i + \Omega_1^{nst}$ which is 1^b-closed w.r.t. T_2^j. Then $S_2^i \neq_{\forall\Pi_1^b} T_2^j$.*

Proof. Under the assumption the last Theorem shows that S_2^i does not prove $\mathrm{RCon}^*(T_2^j)$, while T_2^j does prove it (cf. Theorem 12). $\square$

This proves 3. $\Rightarrow$ 1. of Theorem 4.

Corollary 21. *There is no model M of $T_2^i + \Omega_1^{nst}$ which is 1^b-closed w.r.t. T_2^i.*

Proof. Otherwise we would get $M \models \neg\,\mathrm{RCon}^*(T_2^i)$ by Theorem 19, which contradicts $M \models T_2^i$ as $T_2^i \vdash \mathrm{RCon}^*(T_2^i)$. $\square$

The author conjectures that there also is no model M of $S_2^i + \Omega_1^{\mathrm{nst}}$ which is 1^b-closed w.r.t. S_2^i.

5 Weak end extensions

Up to now our restricted consistency notion yielded Δ_0^b-elementary extensions. We will change this so that weak end extensions are obtained. To this end we extend our formal language to $\mathcal{L}^{\mathrm{ex}}$ adding uniform small conjunctions and disjunctions:

$$\varphi(x) \in \mathcal{L}^{\mathrm{ex}} \Rightarrow \bigvee_{b \leq |a|} \varphi(I_b),\ \bigwedge_{b \leq |a|} \varphi(I_b) \in \mathcal{L}^{\mathrm{ex}}.$$

The formal Tait-style rules for deriving them are given by

$$(\textstyle\bigvee) \qquad \Delta, \varphi(I_b) \text{ for some } b \leq |a| \Rightarrow \Delta, \bigvee_{b \leq |a|} \varphi(I_b)$$

$$(\textstyle\bigwedge) \qquad \Delta, \varphi(I_b) \text{ for all } b \leq |a| \Rightarrow \Delta, \bigwedge_{b \leq |a|} \varphi(I_b).$$

We define $\Sigma_0^{b,ex}$, $s\Sigma_i^{b,ex}$ etc. analogous to Δ_0^b, $s\Sigma_i^b$ etc. counting small conjunctions and small disjunctions as sharply bounded quantifiers. Now we define $\mathrm{RProof}^*_{T_2^i+WEnd}$ similar to $\mathrm{RProof}^*_{T_2^i}$ in the language $\mathcal{L}^{\mathrm{ex}}$ with additional rules $(\bigvee)$, $(\bigwedge)$ and an additional axiom schema

$$\forall x \leq |I_a| \bigvee_{b \leq |a|} x = I_b$$

for all a. Furthermore cuts are extended to $s\Sigma_i^{b,ex} \cup s\Pi_i^{b,ex}$-formulas. Induction need not to be extended to $s\Sigma_i^{b,ex}$.

We have

Theorem 22. $T_2^i \vdash \mathrm{RCon}^*(T_2^i + WEnd)$.

Proof. There is a polytime transformation $\mathrm{elex} : \mathcal{L}^{\mathrm{ex}} \to \mathcal{L}$ given by

$$\psi \equiv \bigvee_{b\leq|a|} \varphi(I_b) \quad\mapsto\quad \exists x_\psi \leq |I_a| \varphi(x_\psi)$$
$$\psi \equiv \bigwedge_{b\leq|a|} \varphi(I_b) \quad\mapsto\quad \forall x_\psi \leq |I_a| \varphi(x_\psi)$$

which can be formalized in S_2^1. We have

$$\mathrm{elex}(\Sigma_0^{b,ex}) = \Sigma_0^b \qquad \mathrm{elex}(s\Sigma_i^{b,ex}) = s\Sigma_i^b \qquad \text{etc.}$$

Furthermore

$$\mathrm{elex}\big(\forall x \leq |I_a| \bigvee_{b\leq|a|} x = I_b\big) \quad\equiv\quad \forall x \leq |I_a| \exists y \leq |I_a| x = y$$

and

$$x \neq I_0 \wedge \ldots \wedge x \neq I_{|a|} \to x \not\leq I_{|a|}$$

have simple proofs in T_2^0 of size linear in $|a|$ resp. $|a|^2$. Hence

$$S_2^1 \vdash \mathrm{RProv}^*_{T_2^i+WEnd}(\ulcorner\Delta\urcorner) \to \mathrm{RProv}^*_{T_2^i}(\ulcorner\mathrm{elex}(\Delta)\urcorner)$$

and therefore

$$S_2^1 \vdash \mathrm{RCon}^*(T_2^i) \to \mathrm{RCon}^*(T_2^i + WEnd).$$

Now Theorem 12, $T_2^i \vdash \mathrm{RCon}^*(T_2^i)$, yields the assertion. □

Theorem 23. *Assume M is a countable model of $S_2^1 + BL\Sigma_1^b$ without weak end extensions to models of T_2^i. Then $M \vDash \neg\,\mathrm{RCon}^*(T_2^i + WEnd)$.*

Proof. Suppose for the sake of contradiction that $M \vDash \mathrm{RCon}^*(T_2^i + WEnd)$. We extend our forcing construction from the proof of Theorem 11. Again we start extending our languages $\mathcal{L}, \mathcal{L}^{\mathrm{ex}}$ to include witnesses $(d_n)_{n\in\omega}$ (we call them $\mathcal{L}^+$ resp. $\mathcal{L}^{\mathrm{ex}+}$). We call the formalized restricted proof predicate in the language $\mathcal{L}^{\mathrm{ex}+}$ again $\mathrm{RProof}^*_{T_2^i+WEnd}$, obtaining

$$M \vDash \neg\,\mathrm{RProv}^*_{T_2^i+WEnd}(\ulcorner\emptyset\urcorner).$$

We define an increasing chain $(T_n)_{n\in\omega}$ of finite sets of $\mathcal{L}^+$-sentences, such that for every n

$$M \vDash \neg\,\mathrm{RProv}^*_{T_2^i+WEnd}(\ulcorner\neg T_n\urcorner). \tag{5}$$

In the end their union $T^+ := \bigcup_n T_n$ will be a Hintikka set for $\mathcal{L}^+$, and the canonical model N of the atomic sentences in T^+ will be a weak end extension of M and a model of T_2^i, contradicting our assumption.

To ensure that T^+ will have these properties, we carry out several tasks as we build the chain. Beside tasks $(T2)$-$(T3)$, $(H1)$-$(H4)$ described in the proof of Theorem 11 we need also the following task:

$(T1')_{a,t}$ (for $a \in M, t \in \mathcal{L}^+$ closed) If $(t \le |I_a|) \in T^+$, then there is some $b \in M$ such that $(t = I_b) \in T^+$.

If $(T1')$ is fulfilled, then N is a weak end extension of M. To see this, let $a \in M$ and $b \in N$ such that $N \models b \le |a|$, then we have to show $b \in M$. As N is the canonical model, there is some closed $t \in \mathcal{L}^+$ such that $N \models b = t$, hence $N \models t \le |a|$. T^+ is $s\Sigma_i^b$-complete (by $(T2)$), hence $(t \le |I_a|) \in T^+$. Now $(T1')_{a,t}$ yields the existence of some $e \in M$ such that $(t = I_e) \in T^+$, hence $N \models t = e$ and therefore $b = e \in M$.

It remains to show that $(T1')_{a,t}$ is enforceable. Therefore we need to describe an expert handling this task. Let $X \subset \omega$ be her subset and assume $n \in X$. Let her list $\{(a,t) : a \in M, t \in \mathcal{L}^+ \text{ closed}\}$ using her set X. Of course here we need that M is countable. Assume that (a,t) is the element indexed by n in her list.

If $(t \le |I_a|) \notin T_{n-1}$ let $T_n := T_{n-1}$. Otherwise $(t \le |I_a|) \in T_{n-1}$. Suppose for the sake of contradiction that

$$M \models \forall b \le |a| \exists p \, \mathrm{RProof}^*_{T_2^i + WEnd}(p, \ulcorner \neg T_{n-1}, t \ne I_b \urcorner).$$

As M is a model of $S_2^1 + BL\Sigma_1^b$ we obtain

$$M \models \exists P \forall b \le |a| \, \mathrm{RProof}^*_{T_2^i + WEnd}((P)_b, \ulcorner \neg T_{n-1}, t \ne I_b \urcorner),$$

hence by $(\bigwedge)$

$$M \models \mathrm{RProv}^*_{T_2^i + WEnd}(\ulcorner \neg T_{n-1}, \bigwedge_{b \le |a|} t \ne I_b \urcorner).$$

As $(t \le |I_a|) \in T_{n-1}$ we get

$$M \models \mathrm{RProv}^*_{T_2^i + WEnd}(\ulcorner \neg T_{n-1}, t \le |I_a| \urcorner),$$

hence

$$M \models \mathrm{RProv}^*_{T_2^i + WEnd}(\ulcorner \neg T_{n-1}, \exists x \le |I_a| \bigwedge_{b \le |a|} x \ne I_b \urcorner)$$

thus $M \models \mathrm{RProv}^*_{T_2^i + WEnd}(\ulcorner \neg T_{n-1} \urcorner)$ by a $\Sigma_0^{b,ex}$-cut contradicting (5) for $n-1$. Hence there is some $b \le |a|$ in M such that

$$M \models \neg \, \mathrm{RProv}^*_{T_2^i + WEnd}(\ulcorner \neg T_{n-1}, t \ne I_b \urcorner).$$

Let $T_n := T_{n-1} \cup \{t \ne I_b\}$.

Using this strategy she fulfills her task as X is infinite. □

Corollary 24. $(1 \le i \le j)$ *Assume that there is a countable model of* $S_2^i + BL\Sigma_1^b$ *without weak end extensions to models of* T_2^j*. Then* $S_2^i \ne_{\forall \Pi_1^b} T_2^j$*.*

Proof. Under the assumption the last Theorem shows that S_2^i does not prove $\mathrm{RCon}^*(T_2^j + WEnd)$, while T_2^j does prove it (cf. Theorem 22). □

This proves 4. ⇒ 1. of Theorem 4.

6 Extending results

First let us remark that if we modify our restricted provability notion to include true $s\Sigma_k^b \cup s\Pi_k^b$-formulas as axioms we would characterize the $\forall\Pi_k^b$-separation of S_2^i from T_2^j. Also it is obvious how to adjust the coding $\ulcorner . \urcorner^*$ to include functions with stronger but sub-exponential growth rates like $\#_3, \#_4, \ldots$.

Now let τ be an $\forall(\Sigma_i^b \cup \Pi_i^b)$-sentence. We are going to argue that the previously obtained results can be extended by adding τ to the theories.

W.l.o.g. let $\tau = \forall y \tau'(y)$ for some $\tau' \in s\Sigma_i^b \cup s\Pi_i^b$, because we are adding τ to T_2^i, and T_2^i knows $\Sigma_i^b = s\Sigma_i^b$. We extend $\mathrm{RProof}^*_{T_2^i}$ to $\mathrm{RProof}^*_{T_2^i+\tau}$ by allowing all instances of $\forall y \tau'(y)$ as additional axioms, i.e. axioms of the form $\tau'(t)$ for any $\mathcal{L}$-term t. Then $\mathrm{RProv}^*_{T_2^i+\tau}$ and $\mathrm{RCon}^*(T_2^i + \tau)$ are defined in the obvious way.

Theorem 25. $T_2^i + \tau \vdash \mathrm{RCon}^*(T_2^i + \tau)$.

Proof. See also Theorem 12.

S_2^1 proves Tarski's conditions for all formulas in a restricted-$T_2^i + \tau$-proof P of the emptyset, therefore we have for all axioms $\tau'(t)$ occuring in $P = \langle p, \vec{b}, \vec{t}, \vec{d}, \vec{T}, \vec{D} \rangle$

$$\forall y \tau'(y) \to \forall b_0 \le t_0 \ldots \forall b_k \le t_k \, \mathrm{Tr}_i(\ulcorner \tau'(t) \urcorner, \vec{b}).$$

Thus the argument runs the same way as in the proof of Theorem 12, i.e. we can show

$$T_2^i \vdash \forall y \tau'(y) \to \mathrm{RCon}^*(T_2^i + \tau).$$

□

Again we have that $\mathrm{RCon}^*(T_2^i + \tau)$ is the strongest $\forall\Pi_1^b$-statement (over S_2^1) provable in $T_2^i + \tau$.

Corollary 26. *For all $\varphi \in \forall\Pi_1^b$ such that $T_2^i + \tau \vdash \varphi$ we have $S_2^1 \vdash \mathrm{RCon}^*(T_2^i + \tau) \to \varphi$.* □

Theorem 27. *Assume M is a countable model of $S_2^1 + BL\Sigma_1^b$ without weak end extensions to models of $T_2^i + \tau$. Then $M \models \neg\,\mathrm{RCon}^*(T_2^i + \tau + WEnd)$.*

Proof. Assuming $M \models \mathrm{RCon}^*(T_2^i + \tau + WEnd)$ we modify our forcing construction obtaining a Hintikka set T^+ such that

$$\tau'(t) \in T^+ \quad \text{for all closed } \mathcal{L}^+\text{-terms } t.$$

Then the canonical model N generated from T^+ fulfills $N \models \forall y \tau'(y)$, thus $N \models T_2^i + \tau$ and N is a weak end extension of M contradicting our assumption. □

Similarily we can prove the following Theorems.

Theorem 28. *Assume M is a model of S_2^1 which does not have a log^3-proper Δ_0^b-elementary extension to a model of $T_2^i + \tau$. Then $M \models \neg\,\mathrm{RCon}^*(T_2^i + \tau)$.* □

Theorem 29. *Assume M is a model of $S_2^1 + \Omega_1^{nst}$ which is 1^b-closed w.r.t. $T_2^i + \tau$. Then $M \models \neg\,\mathrm{RCon}^*(T_2^i + \tau)$.* □

Hence we obtain

Corollary 30. *Let $1 \leq i \leq j$. The following are equivalent:*

1. *$S_2^i + \tau$ is $\forall\Pi_1^b$-separated from $T_2^j + \tau$.*
2. *There is a model of $S_2^i + \tau$ which does not have a log^3-proper Δ_0^b-elementary extension to a model of $T_2^j + \tau$.*
3. *There is a model of $S_2^i + \tau + \Omega_1^{nst}$ which is 1^b-closed w.r.t. $T_2^j + \tau$.*
4. *There is a countable model of $S_2^i + \tau + BL\Sigma_1^b$ without weak end extensions to models of $T_2^j + \tau$.* □

Remark 31. *Considering Ω_1^{nst} we can extend the results allowing τ to be an $\exists\forall(\Sigma_i^b \cup \Pi_i^b)$-sentences which has as a parameter the nonstandard element given by Ω_1^{nst}. I.e., assume that our language $\mathfrak{L}$ is extended by a new constant c. Let $\tilde{\Omega}_1^{nst}(c)$ be the following $\mathfrak{L}_{\omega_1\omega}$-sentence:*

$$\tilde{\Omega}_1^{nst}(c) := \big(\bigwedge_{k\in\omega} (\underline{k} < c)\big) \wedge \forall x \exists y (||x|| \cdot c = ||y||).$$

Thus $\Omega_1^{nst} = \exists z \tilde{\Omega}_1^{nst}(z)$. Then we can consider $\tau(c)$ instead of τ in the previously obtained results where Ω_1^{nst} is replaced by $\tilde{\Omega}_1^{nst}(c)$.

7 Towards proper weak end extensions

In section 5 we have connected the $\forall\Pi_1^b$-separation of S_2^i from T_2^j with models of $S_2^i + BL\Sigma_1^b$ which do not have weak end extensions to models of T_2^j. But we are really interested in a connection to models without *proper* weak end extensions, because there is a similar (open) question for models of $I\Delta_0$: are there models of $I\Delta_0 + B\Sigma_1$ without proper end extensions to models of $I\Delta_0$? Furthermore, there exists a Π_1-sentence τ (a version of the Tableau consistency of $I\Delta_0$) and a model of $I\Delta_0 + \Omega_1 + \tau + B\Sigma_1$ which has no proper end extensions to models of $I\Delta_0 + \Omega_1 + \tau$ (cf. [1]).

Up to now we have not achieved a connection to proper weak end extensions. We will describe two possible ways in this direction now.

Wilkie and Paris in [22] defined a Π_3-sentence ψ such that

$$I\Delta_0 + B\Sigma_1 \qquad I\Delta_0 + B\Sigma_1 + \psi \qquad I\Delta_0 + B\Sigma_1 + \neg\psi$$

all have the same Π_1-consequences. This can be improved to

Proposition 32. *There is a $\forall\exists\Sigma_1^b$-sentence χ such that for $S_2^1 \subseteq T \subseteq S_2$*

$$T \qquad T + \chi \qquad T + \neg\chi$$

all have the same Π_1-consequences.

Proof. Let $\mathrm{Log}_a(z) := \mu u.z \leq 2^{|a|^u}$, then $\mathrm{Log}_a(z) = u$ has a Δ_1^b-description in S_2^1. Let χ be the sentence equivalent to

$$\forall a \forall z (2^{|a|^{2^{\mathrm{Log}_a(z)}}} \text{ exists}).$$

Then

$$T \qquad\qquad\qquad T+\chi \qquad\qquad\qquad T+\neg\chi$$

all have the same Π_1-consequences. To see this, suppose

$$T+\neg\chi \vdash \forall x\theta(x), \quad \theta\in\Delta_0$$

but

$$T \nvdash \forall x\theta(x).$$

Then we find a model $K \vDash T+\neg\theta(a)$ such that a^{η^t} exists in K for some $\mathbb{N}<\eta, t<a$. Let

$$L^K(a,\eta) := \{x\in K \,:\, x\leq 2^{|a|^{\eta^n}} \text{ some } n\in\mathbb{N}\} \subseteq_e K$$

then $L^K(a,\eta)\vDash T+\neg\theta(a)$ (this is true for arbitrary η). Furthermore, $L^K(a,\eta)\vDash \neg\chi$, because $b:=2^{|a|^\eta}\in L^K(a,\eta)$ and $\mathrm{Log}_a(b)=\eta$, but $2^{|a|^{2^\eta}} \not\leq 2^{|a|^{\eta^n}}$ for all $n\in\mathbb{N}$.

The case for χ is similar, taking 2 in place of η and observing $L^K(a,2)\vDash \chi$. ☐

Remark 33. *The same χ from the proof of the last proposition also fulfills that*

$$T+BL\Sigma_1^b \qquad\qquad T+BL\Sigma_1^b+\chi \qquad\qquad T+BL\Sigma_1^b+\neg\chi$$

have the same Π_1-consequences. This can be seen by adapting Theorem 1 from [22] in the form

$$M\subset_e K\vDash S_2^1 \quad\Rightarrow\quad M\vDash BL\Sigma_1^b.$$

Open Problem 1. *Can this be improved to finding a sentence ϕ such that Proposition 32 holds for ϕ instead of χ and such that the formalized proof predicate can be extended to ϕ and $\neg\phi$ fulfilling*

$$T_2^j+\phi\vdash \mathrm{RCon}^*(T_2^j+\phi) \qquad\qquad T_2^j+\neg\phi\vdash \mathrm{RCon}^*(T_2^j+\neg\phi)\ ?$$

Having this, we would get

$$T_2^j\vdash \mathrm{RCon}^*(T_2^j+\phi)\wedge \mathrm{RCon}^*(T_2^j+\neg\phi)$$

using Proposition 32.

On the other hand we would be able to show that if M is a model of $S_2^1+BL\Sigma_1^b$ without proper weak end extensions to models of T_2^j, then

$$M\vDash \neg\,\mathrm{RCon}^*(T_2^j+\phi)\vee\neg\,\mathrm{RCon}^*(T_2^j+\neg\phi).$$

To see this observe $M\vDash\phi\vee\neg\phi$. W.l.o.g. we may assume $M\vDash\phi$. Now assuming $M\vDash \mathrm{RCon}^*(T_2^j+\neg\phi)$ would produce a weak end extension N of M which is a model of $T_2^j+\neg\phi$. But then $M\neq N$ as $M\vDash\phi$ and $N\vDash\neg\phi$, hence the extension is a proper one contradicting our assumption. Hence

$$S_2^i\nvdash \mathrm{RCon}^*(T_2^j+\phi)\wedge \mathrm{RCon}^*(T_2^j+\neg\phi).$$

The second possibility bases on an ultrapower construction described by Buss in [7]. Suppose

$$M \models S_2^i + BL\Pi_i^b + BB\Pi_i^b$$

where $BB\Pi_i^b$ is the sharply bounded replacement schema

$$(\forall x \leq |a|)(\exists y \leq b)\varphi(x,y) \to (\exists w)(\forall x \leq |a|)\varphi(x,(w)_x)$$

for $\varphi \in \Pi_i^b$. Note that by results of Buss [8] and Ressayre [19] the theory $S_2^i + BL\Pi_i^b + BB\Pi_i^b$ is $\forall\Sigma_{i+1}^b$-conservative over S_2^i. Buss shows in [7] that there is a proper $(\Sigma_i^b \cap \Pi_i^b)$-elementary weak end extension N of M such that $\log N = \log M$ and $N \models T_2^{i-1}$.

Open Problem 2. *Can this be improved such that $N \models BL\Sigma_1^b$ and N is a $\forall\Sigma_1^b$-elementary extension of M?*

Then we could argue as follows: If M does not have proper weak end extensions to models of T_2^j, then $M \models \neg\,\mathrm{RCon}^*(T_2^j)$. Because assuming $M \models \mathrm{RCon}^*(T_2^j)$ would imply $N \models \mathrm{RCon}^*(T_2^j)$, and our construction from section 5 would yield a weak end extensions of N to a model N' of T_2^j, but N' would now be a proper weak end extension of M – contradiction.

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