Exact bounds for lengths of reductions in typed λ-calculus

Arnold Beckmann
Institut für mathematische Logik und Grundlagenforschung
der Westfälischen Wilhelms-Universität Münster
Einsteinstr. 62, D-48149 Münster, Germany

January 29, 2003

Abstract

We determine the exact bounds for the length of an arbitrary reduction sequence of a term in the typed λ-calculus with β-, ξ- and η-conversion. There will be two essentially different classifications, one depending on the height and the degree of the term and the other depending on the length and the degree of the term.

Although it is well known that the full reduction tree for any term of the typed λ-calculus – and thus also any reduction sequence of that term – is finite, there exists – to the authors best knowledge – a gap concerning the classification of their growth rates. We only know that the growth rates are in $\mathcal{E}^4$ (cf. Schwichtenberg [S82]). A better upper bound is achieved in [S91]. There the estimate depends on the degree $\mathrm{g}(r)$, the height $\mathrm{h}(r)$ and the arities of free variables $\mathrm{ar}(r)$ of a term r. It is shown that any reduction sequence for r is bounded by

$$\mathrm{ar}(r)^{2_{\mathrm{g}(r)}(\mathrm{g}(r)+2\,\mathrm{h}(r)+2\,\mathrm{ar}(r)+2)}$$

where $2_m(n)$ is recursively defined by $2_0(n) = n$ and $2_{m+1}(n) = 2^{2_m(n)}$. In this paper we will show that this bound can be improved to

$$2_{\mathrm{g}(r)+1}(\mathrm{h}(r)) \qquad \text{resp.} \qquad 2_{\mathrm{g}(r)}(\mathrm{l}(r))$$

where $\mathrm{l}(r)$ denotes the length of r. Together with an optimal lower bound this closes the gap.

1 Introduction

Let r, s, t denote terms of the typed λ-calculus.[1] The *length* $\mathrm{l}(r)$ and the *height* $\mathrm{h}(r)$ of r are defined by $\mathrm{l}(x) = 1$, $\mathrm{l}(\lambda xr) = \mathrm{l}(r) + 1$, $\mathrm{l}(rs) = \mathrm{l}(r) + \mathrm{l}(s)$ and

[1] For a general definition of the typed λ-calculus see Barendregt [B84].

$\mathrm{h}(x) = 0$, $\mathrm{h}(\lambda x r) = \mathrm{h}(r) + 1$, $\mathrm{h}(rs) = \max(\mathrm{h}(r), \mathrm{h}(s)) + 1$. By induction on r we immediately see $\mathrm{l}(r) \leq 2^{\mathrm{h}(r)}$. A ground type ι has *level* $\mathrm{lv}(\iota) = 0$ and $\mathrm{lv}(\rho \to \sigma) = \max(\mathrm{lv}(\rho) + 1, \mathrm{lv}(\sigma))$. The *level* $\mathrm{lv}(r)$ of r is defined to be the level $\mathrm{lv}(\sigma)$ of its type σ, the *degree* $\mathrm{g}(r)$ of r is defined to be the maximum of the levels of subterms of r.

With $\mathrm{d}(r)$ we denote the maximum of lengths of reduction sequences for r with respect to $\longrightarrow^1$, the one step reduction using β-, ξ- and η-conversion rules. Our investigations will focus on the following functions estimating derivation lengths:

$$\mathrm{dl}_n(N) := \max\{\mathrm{d}(r) \, : \, r \text{ a term}, \mathrm{g}(r) \leq n, \mathrm{l}(r) \leq N\} \text{ and}$$
$$\mathrm{dh}_n(N) := \max\{\mathrm{d}(r) \, : \, r \text{ a term}, \mathrm{g}(r) \leq n, \mathrm{h}(r) \leq N\}.$$

We introduce some common notions for comparing growth rates of functions. The symbols $f(n) = \mathrm{O}(g(n))$, $f(n) = \Omega(g(n))$ and $f(n) = \Theta(g(n))$ denote that eventually $f(n) \leq c \cdot g(n)$, $f(n) \geq c \cdot g(n)$ and $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ where c, c_1, c_2 are positive constants. Obviously $f(n) = \Theta(g(n))$ iff $f(n) = \mathrm{O}(g(n))$ and $f(n) = \Omega(g(n))$. For S one of $\mathrm{O}, \Omega, \Theta$ the symbols $f(n) = h(S(g(n)))$ denote that there is a functions $\phi(n)$ such that $f(n) = h(\phi(n))$ and $\phi(n) = S(g(n))$.

The main result in this paper will be the following

Main Theorem *Independent of n we have*

$$\mathrm{dh}_n(N) = 2_{n+1}(\Theta(N)) \quad \text{and} \quad \mathrm{dl}_n(N) = 2_n(\Theta(N)).$$

More exact we will prove the Main Theorem in the following form: There are strictly positive constants c_l, c_h such that for all n there exists some $N_0(n)$ such that for all $N \geq N_0(n)$

$$\begin{aligned} 2_n(\frac{N}{c_l}) \leq \quad & \mathrm{dl}_n(N) \quad \leq 2_n(N) \\ 2_{n+1}(\frac{N}{c_h}) \leq \quad & \mathrm{dh}_n(N) \quad \leq 2_{n+1}(N) \end{aligned}$$

Remark Our investigations are concerned with *worst case* reductions, i.e. longest possible reduction chains. If one considers arbitrary reduction chains the following super-exponential lower bounds are known. In §1 of [S82] terms of length $O(n)$ and degree $n+2$ are defined such that *every* reduction sequence up to the normal form has length $\geq 2_{n-2}(1) - n$. Applying the same argument from §1 of [S82] to our lower bounds A^n_N shows that there are terms of length $O(N)$ and degree $n+1$ such that *every* reduction sequence up to the normal form has length $\geq 2_{n-1}(N) - N$ for N big enough. It would be interesting to know what are the exact classifications of dl^*_n, dh^*_n, where dl^*_n and dh^*_n are obtained from dl_n and dh_n resp. by replacing $\mathrm{d}(r)$ with the *minimum* of lengths of reduction sequences for r with respect to $\longrightarrow^1$.

Remark The author learned from the referee that the rules defining our "expanded head reduction tree" were already considered by van Raamsdonk and Severi in [R95] as well as Goguen in [G94]. Furthermore, Loader independently claimed slightly weaker upper bounds for $\mathrm{dl}_n(N)$ (i.e., $2_n(K \cdot N)$ for some $K > 1$) in an unpublished report [L98].

2 Upper bounds

In [S91] it is proved that $\mathrm{d}(r) \leq \mathrm{ar}(r)^{2_{\mathrm{g}(r)}(c \cdot (\mathrm{h}(r)))}$ for some $c > 0$ where $\mathrm{ar}(r)$ is the maximum of the arities of free variables of r. Unfortunately the arity of a term of type 0 cannot be bounded uniformly by a function in the length or height of the term. E.g. let x_n, y_n be variables of type $(0^n \to 0) \to 0$ resp. $0^n \to 0$, then $\mathrm{ar}(x_n y_n) = n$, $\mathrm{l}(x_n y_n) = 2$ and $\mathrm{h}(x_n y_n) = 1$ for all n.

In showing $\mathrm{dl}_n(N) \leq 2_n(N)$ we do obtain also the exact bounds depending on the heights, because $\mathrm{l}(r) \leq 2^{\mathrm{h}(r)}$ yields

$$\mathrm{dh}_n(N) \leq \mathrm{dl}_n(2^N) \leq 2_n(2^N) = 2_{n+1}(N). \tag{1}$$

On the other hand we cannot argue that $\mathrm{dh}_n(N) \leq 2_{n+1}(N)$ implies $\mathrm{dl}_n(N) \leq 2_n(N)$, because there are terms r of arbitrary length such that $\mathrm{l}(r) = \mathrm{h}(r) + 1$, e.g. $r = \lambda \vec{x}.y$.

The key observation in [S91] is that the number of nodes with conversion in the head reduction tree of a λ-I-term bounds the length of any reduction sequence. There, λ-I-terms are special λ-terms having the property of never forgetting subterms in reductions. More formally a term r is called a λ-I-term if for any subterm of the form λxs one has $x \in \mathrm{fvar}(s)$, where $\mathrm{fvar}(s)$ is the set of variables free in s. The general case can be reduced to the case of λ-I-terms by introducing variants of a term. Such a variant r° of a term r is a λ-I-term with $\mathrm{g}(r^\circ) = \mathrm{g}(r)$ and $\mathrm{d}(r^\circ) \geq \mathrm{d}(r)$, but only $\mathrm{l}(r^\circ) = \mathrm{O}(\mathrm{l}(r)^2)$. Thus we cannot use this approach directly to obtain the desired bound.

In our approach we consider an expanded head reduction tree. Each node labeled with a β-redex $(\lambda xr)s$ will have two childs, $r[x := s]$ and s. Thus, also in the case $x \notin \mathrm{fvar}(r)$ the expanded head reduction tree will control conversions in s. Hence the number of nodes with conversion in the expanded head reduction tree bounds the length of any reduction sequence, this time for arbitrary typed λ-terms.

The main difference of our calculus $\vdash^{\alpha}_{\rho} r$ compared with the calculus in [S91] is beside the refinement of the β-Rule, that λ-terms of arbitrary level are derived, and that the width of the expanded head reduction tree is also controled. The latter yields a later Estimate Lemma which is independent from the arities of certain free variables.

The techniques used here are refinements and further developments of those presented in [S91].

Definition *We define* $\vdash^{\alpha}_{\rho} r$ *for* λ*-terms* r *of arbitrary level and* $\alpha, \rho < \omega$ *inductively by*

(β-Rule) *If* $\mid\!\frac{\alpha}{\rho}\, r[x := s]\vec{t}$ *and* $\mid\!\frac{\alpha}{\rho}\, s$*, then* $\mid\!\frac{\alpha+1}{\rho}\, (\lambda xr)s\vec{t}$*.*

(β_0-Rule) *If* $\mid\!\frac{\alpha}{\rho}\, r$ *then* $\mid\!\frac{\alpha+1}{\rho}\, \lambda xr$*.*

(Variable Rule) *If* $\mid\!\frac{\alpha}{\rho}\, t_i$ *for* $i = 1, \ldots, n$*, then* $\mid\!\frac{\alpha+n}{\rho}\, x\vec{t}$*. In particular* $\mid\!\frac{\alpha}{\rho}\, x$ *for any variable* x *and* $\alpha, \rho < \omega$*.*

(Cut Rule) *If* $\mid\!\frac{\alpha}{\rho}\, r$*,* $\mathrm{lv}(r) \leq \rho$ *and* $\mid\!\frac{\alpha}{\rho}\, t$*, then* $\mid\!\frac{\alpha+1}{\rho}\, rt$*.*

The calculus allows a structural rule, i.e. if $\mid\!\frac{\alpha}{\rho}\, r$ *and* $\alpha \leq \alpha' < \omega$*,* $\rho \leq \rho' < \omega$*, then* $\mid\!\frac{\alpha'}{\rho'}\, r$*.*

First we observe that $\mid\!\frac{\alpha}{0}\, r$ can be viewed as a tree which is generated in a unique way. We call this tree (with the α's stripped off) the expanded head reduction tree. It has the desired property that $\#r$, the number of nodes with conversion in it, bounds the length on any reduction sequence of r. More precisely we define by induction on $\mid\!\frac{\alpha}{0}\, r$:

$$\begin{aligned} \#\left((\lambda xr)s\vec{t}\right) &:= \#\left(r[x := s]\vec{t}\right) + 1 + \#s \\ \#\left(\lambda xr\right) &:= \#r + 1 \\ \#\left(xt_1 \ldots t_n\right) &:= \sum_{i=1}^{n} \#t_i. \end{aligned}$$

Before we show that $\#r$ has the desired properties we need some technical lemmas.

Lemma 1 $\#r = \#r[x := y]$.

Proof. The proof by induction on $\mid\!\frac{\alpha}{0}\, r$ is obvious. □

Lemma 2 $\#(ry) \geq \#r$.

Proof. By induction on $\mid\!\frac{\alpha}{0}\, ry$.

$$\begin{aligned} \#\left((\lambda xr)s\vec{t}y\right) &= \#\left(r[x := s]\vec{t}y\right) + 1 + \#s \\ &\geq \#\left(r[x := s]\vec{t}\right) + 1 + \#s = \#\left((\lambda xr)s\vec{t}\right) \\ \#\left((\lambda xr)y\right) &= \#(r[x := y]) + 1 + \#y = \#r + 1 = \#\left(\lambda xr\right) \\ \#\left(xt_1 \ldots t_n y\right) &= \sum_{i=1}^{n} \#t_i + \#y = \sum_{i=1}^{n} \#t_i = \#\left(xt_1 \ldots t_n\right). \end{aligned}$$

We used $\#y = 0$ and Lemma 1. □

Now we are able to prove our

Main Lemma *If* $r \longrightarrow^1 s$*, then* $\#r > \#s$*.*

Proof. We will show a more general assertion. If $z \in \mathrm{fvar}(r)$, then

(β) $\#\left(r[z := (\lambda xp)q]\right) > \#\left(r[z := p[x := q]]\right)$

(η) $\#\,(r[z := (\lambda x.px)]) > \#\,(r[z := p])$ if $x \notin \mathrm{fvar}(p)$.

Let $t^* := t[z := (\lambda xp)q]$ and $t' := t[z := p[x := q]]$ for assertion (β) resp. $t^* := t[z := (\lambda x.px)]$ and $t' := t[z := p]$ for (η). We prove both assertions by induction on $\#r^*$.

$$\begin{aligned}\#\left((\lambda yr)s\vec{t}\right)^* &= \#\left(r[y := s]\vec{t}\right)^* + 1 + \#s^* \\ &> \#\left(r[y := s]\vec{t}\right)' + 1 + \#s' \\ &= \#\left((\lambda yr)s\vec{t}\right)'\end{aligned}$$

For ">" it is important that we have $z \in \mathrm{fvar}(r[y := s]\vec{t})$ or $z \in \mathrm{fvar}(s)$. This is the reason why we formulated the β-Rule in the definition of $\vdash^{\alpha}_{\rho} r$ as we did.

$$\#(\lambda yr)^* = \#r^* + 1 > \#r' + 1 = \#(\lambda yr)'.$$

For the next case we assume $z \neq y$, hence $n > 0$ and $z \in \mathrm{fvar}(\vec{t})$.

$$\#\,(yt_1 \dots t_n)^* = \sum_{i=1}^{n} \#t_i^* > \sum_{i=1}^{n} \#t_i' = \#\,(yt_1 \dots t_n)'.$$

Considering assertion (β) we have

$$\begin{aligned}\#\,(zt_1 \dots t_n)^* &= \#\left((\lambda xp)q\vec{t}^{\,*}\right) \\ &= \#\left(p[x := q]\vec{t}^{\,*}\right) + 1 + \#q \\ &\geq \#\left(p[x := q]\vec{t}^{\,\prime}\right) + 1 \\ &> \#\left(p[x := q]\vec{t}^{\,\prime}\right) \\ &= \#(z\vec{t})'.\end{aligned}$$

For assertion (η) we distinguish two cases.

$$\#z^* = \#(\lambda x.px) = \#(px) + 1 > \#(px) \geq \#p$$

using Lemma 2.

$$\begin{aligned}\#\left(zt_0\vec{t}\right)^* &= \#\left((\lambda x.px)t_0^*\vec{t}^{\,*}\right) \\ &= \#\left(pt_0^*\vec{t}^{\,*}\right) + 1 + \#(t_0^*) \\ &\geq \#\left(pt_0'\vec{t}^{\,\prime}\right) + 1 \\ &> \#\left(pt_0'\vec{t}^{\,\prime}\right) \\ &= \#(zt_0\vec{t})'.\end{aligned}$$

□

We are now going to eliminate cuts.

Renaming Lemma *If* $\vdash^{\alpha}_{\rho} r$*, then* $\vdash^{\alpha}_{\rho} r[x := y]$.

Proof. The proof by induction on $\vdash^{\alpha}_{\rho} r$ is obvious. □

Appending Lemma *If* $\vdash^{\alpha}_{\rho} r$ *and* ry *is a term, then* $\vdash^{\alpha+1}_{\rho} ry$.

Proof. By induction on $\vdash^{\alpha}_{\rho} r$.
β-Rule. By induction hypothesis we have $\vdash^{\alpha+1}_{\rho} r[x := u]\vec{t}y$ and $\vdash^{\alpha}_{\rho} u$, hence $\vdash^{\alpha+2}_{\rho} (\lambda xr)u\vec{t}y$ by the β-Rule.
β_0-Rule. We have $\vdash^{\alpha}_{\rho} r$, hence $\vdash^{\alpha}_{\rho} r[x := y]$ by the Renaming Lemma. Thus $\vdash^{\alpha+1}_{\rho} (\lambda xr)y$ by the β-Rule.
Variable Rule. By assumption we have $\vdash^{\alpha}_{\rho} t_i$. Furthermore $\vdash^{\alpha}_{\rho} y$ by the Variable Rule, hence $\vdash^{\alpha+n+1}_{\rho} xt_1 \dots t_n y$ by the Variable Rule.
Cut Rule. We have $\vdash^{\alpha+1}_{\rho} rt$ and $\mathrm{lv}(r) \leq \rho$, hence $\mathrm{lv}(rt) \leq \mathrm{lv}(r) \leq \rho$. By the Variable Rule we obtain $\vdash^{\alpha+1}_{\rho} y$, thus $\vdash^{\alpha+2}_{\rho} rty$ by a cut. □

Observe that the following Estimate Lemma does not depend on the arity $\mathrm{ar}(r)$ of r.

Estimate Lemma *If* $\vdash^{\alpha}_{0} r$*, then* $\#r \leq 2^{\alpha}$.

Proof. We show $\vdash^{\alpha}_{0} r \Rightarrow \#r \leq 2^{\alpha} - 1$ by induction on $\vdash^{\alpha}_{0} r$.
β-Rule. $\#\left((\lambda xr)s\vec{t}\right) = \#\left(r[x := s]\vec{t}\right) + 1 + \#(s) \leq (2^{\alpha} - 1) + 1 + (2^{\alpha} - 1) \leq 2^{\alpha+1} - 1$
β_0-Rule. $\#(\lambda xr) = \#r + 1 \leq (2^{\alpha} - 1) + 1 \leq 2^{\alpha+1} - 1$
Variable-Rule. $\#(xt_1 \dots t_n) = \sum_{i=1}^{n} \#t_i \leq n \cdot (2^{\alpha} - 1) \leq 2^{\alpha+n} - 1$. □

Substitution Lemma *If* $\vdash^{\alpha}_{\rho} r$ *and* $\vdash^{\beta}_{\rho} s_j$ *and* $\mathrm{lv}(s_j) \leq \rho$*,* $j = 1, \dots, k$*, then* $\vdash^{\beta+\alpha}_{\rho} r[\vec{x} := \vec{s}]$.

Proof. By induction on $\vdash^{\alpha}_{\rho} r$. We write t^* for $t[\vec{x} := \vec{s}]$.
β-Rule. By induction hypothesis we have $\vdash^{\beta+\alpha}_{\rho} r^*[x := u^*]\vec{t}^*$ and $\vdash^{\beta+\alpha}_{\rho} u^*$, hence $\vdash^{\beta+\alpha+1}_{\rho} (\lambda xr^*)u^*\vec{t}^*$ by the β-Rule.
β_0-Rule. By induction hypothesis $\vdash^{\beta+\alpha}_{\rho} r^*$, hence $\vdash^{\beta+\alpha+1}_{\rho} (\lambda xr^*)$ by the β_0-Rule.
Variable Rule. By induction hypothesis we have $\vdash^{\beta+\alpha}_{\rho} t_i^*$, hence $\vdash^{\beta+\alpha+n}_{\rho} xt_1^* \dots t_n^*$ by the Variable Rule. If $x \notin \vec{x}$ then we are done.

Otherwise there is some j with $x = x_j$. The presumptions yield $\vdash^{\beta+\alpha}_{\rho} s_j$ and $\mathrm{lv}(s_j) \leq \rho$, thus we obtain $\vdash^{\beta+\alpha+n}_{\rho} s_j t_1^* \dots t_n^*$ by applying n cuts.
Cut Rule. By induction hypothesis $\vdash^{\beta+\alpha}_{\rho} r^*$ and $\vdash^{\beta+\alpha}_{\rho} t^*$ and $\mathrm{lv}(r^*) \leq \rho$, thus $\vdash^{\beta+\alpha+1}_{\rho} r^*t^*$ again by a cut. □

Cut Elimination Lemma *If* $\vdash^{\alpha}_{\rho+1} r$ *then* $\vdash^{2^\alpha}_{\rho} r$.

Proof. We show $\vdash^{\alpha}_{\rho+1} r \Rightarrow \vdash^{2^\alpha-1}_{\rho} r$ by induction on $\vdash^{\alpha}_{\rho+1} r$.
β-Rule. By induction hypothesis we have $\vdash^{2^\alpha-1}_{\rho} r[x := u]\vec{t}$ and $\vdash^{2^\alpha-1}_{\rho} u$, hence $\vdash^{2^\alpha}_{\rho} (\lambda xr)u\vec{t}$ by the β-Rule and $2^\alpha \leq 2^{\alpha+1} - 1$.
β_0-Rule. By induction hypothesis we have $\vdash^{2^\alpha-1}_{\rho} r$, hence $\vdash^{2^\alpha}_{\rho} \lambda xr$ by the β_0-Rule.
Variable Rule. By induction hypothesis $\vdash^{2^\alpha-1}_{\rho} t_i$, hence $\vdash^{2^\alpha+n-1}_{\rho} xt_1 \dots t_n$ by the Variable Rule and $2^\alpha + n - 1 \leq 2^{\alpha+n} - 1$.
Cut Rule. By induction hypothesis we have $\vdash^{2^\alpha-1}_{\rho} r$, $\vdash^{2^\alpha-1}_{\rho} t$ and $\mathrm{lv}(r) \leq \rho + 1$, hence $\mathrm{lv}(t) \leq \rho$. By the Appending Lemma we obtain $\vdash^{2^\alpha}_{\rho} ry$, thus $\vdash^{2^{\alpha+1}-1}_{\rho} rt$ by the Substitution Lemma. □

We embed terms depending on their length instead of height in order to obtain a sharper bound on the cut degrees.

Embedding Lemma
$\mathrm{g}(r) \leq \rho + 1$ *implies* $\vdash^{\mathrm{l}(r)}_{\rho} r$.

Proof. We show $\mathrm{g}(r) \leq \rho + 1 \Rightarrow \vdash^{\mathrm{l}(r)-1}_{\rho} r$ by induction on r.
Case x. The Variable Rule shows $\vdash^{\mathrm{l}(x)-1}_{\rho} x$.
Case λxr. By induction hypothesis $\vdash^{\mathrm{l}(r)-1}_{\rho} r$, hence $\vdash^{\mathrm{l}(\lambda xr)-1}_{\rho} (\lambda xr)$ by the β_0-Rule.
Case ts. By induction hypothesis $\vdash^{\mathrm{l}(t)-1}_{\rho} t$ and $\vdash^{\mathrm{l}(s)-1}_{\rho} s$, thus the Appending Lemma yields $\vdash^{\mathrm{l}(t)}_{\rho} ty$. Since $\mathrm{lv}(t) \leq \rho+1$ we have $\mathrm{lv}(s) \leq \rho$, hence $\vdash^{\mathrm{l}(t)+\mathrm{l}(s)-1}_{\rho} ts$ by the Substitution Lemma. □

With the Embedding Lemma and the Cut Elimination Lemma it follows that the expanded head reduction tree of r with $\mathrm{g}(r) > 0$ has the

$$\text{height } \leq 2_{\mathrm{g}(r)-1}(\mathrm{l}(r)).$$

The Estimate Lemma now shows

$$\#r \leq 2^{2_{\mathrm{g}(r)-1}(\mathrm{l}(r))} = 2_{\mathrm{g}(r)}(\mathrm{l}(r))$$

Together with the Main Lemma this yields

$$\mathrm{d}(r) \leq \#r \leq 2_{\mathrm{g}(r)}(\mathrm{l}(r)).$$

Hence we obtain for $n > 0$

$$\mathrm{dl}_n(N) \leq 2_n(N). \tag{2}$$

This is also true for $n = 0$ because the only terms t with $\mathrm{g}(t) = 0$ are variables $t = x$, hence $\mathrm{dl}_0(N) = 0$.

Together with observation (1) from the beginning of this section (2) also shows

$$\mathrm{dh}_n(N) \leq 2_{n+1}(N). \tag{3}$$

Remarks *The techniques from the last part can also be applied to reductions in Combinatory Logic with combinators K and S which yield the same upper bound.*

3 Lower bounds

We are going to define terms A_N^n and B_N^n such that

1. $\mathrm{g}(A_N^n) = \mathrm{g}(B_N^n) = n+1$,
2. $\mathrm{d}(A_N^n) \geq 2_{n+1}(N)$ and $\mathrm{d}(B_N^n) \geq 2_{n+2}(N)$,
3. $\mathrm{l}(A_N^n) = \mathrm{O}(N) = \mathrm{h}(B_N^n)$ independent of n.

This yields

$$\mathrm{dl}_n(N) = 2_n(\Omega(N)) \quad \text{and} \quad \mathrm{dh}_n(N) = 2_{n+1}(\Omega(N)) \tag{4}$$

because

1. $\mathrm{l}(A_N^n) \leq c \cdot N$ for some $c > 0$ and N big, hence

$$\mathrm{dl}_{n+1}(N) \geq \mathrm{d}(A^n_{\lfloor \frac{N}{c} \rfloor}) \geq 2_{n+1}(\lfloor \frac{N}{c} \rfloor) \geq 2_{n+1}(\frac{N}{c+1})$$

   for $N \geq c \cdot (c+1)$ as $\lfloor \frac{N}{c} \rfloor \geq \frac{N}{c} - 1 \geq \frac{N}{c} - \frac{N}{c \cdot (c+1)} = \frac{N}{c+1}$.

2. $\mathrm{h}(B_N^n) \leq c \cdot N$ for some $c > 0$ and N big, hence

$$\mathrm{dh}_{n+1}(N) \geq \mathrm{d}(B^n_{\lfloor \frac{N}{c} \rfloor}) \geq 2_{n+2}(\lfloor \frac{N}{c} \rfloor) \geq 2_{n+2}(\frac{N}{c+1})$$

   for $N \geq c \cdot (c+1)$.

Fix some ground type 0. For natural numbers n define a type $o(n)$ via $o(0) = 0$ and $o(n+1) = o(n) \to o(n)$, then $\mathrm{lv}(o(n)) = n$. With $[u]^k(v)$ we denote the k-fold iteration of u applied to v, i.e. $[u]^k(v) = \underbrace{u(\ldots(u}_{k-\text{times } u}\, v)\ldots)$. We define the generalized CHURCH-numerals for a type σ by

$$\overline{N}^\sigma = \lambda f^{\sigma \to \sigma} \lambda x^\sigma [f]^N(x).$$

Then $\overline{N}^\sigma$ is of type $(\sigma \to \sigma) \to \sigma \to \sigma$ and $\mathrm{g}(\overline{N}^\sigma) = \mathrm{lv}(\overline{N}^\sigma) = \mathrm{lv}(\sigma) + 2$.

Furthermore we fix some variable 0 of type 0 and some variable d of type $0 \to 0 \to 0$. Let $I = \lambda x^0.x$ and $D = \lambda x^0.dxx$. We define the tree-like term $T_k(u)$ of height k via $T_0(u) = u$ and $T_{k+1}(u) = d(T_k(u))(T_k(u))$.

Lemma *Let k be a natural number.*

1. $\mathrm{l}([r]^k(s)) = k \cdot \mathrm{l}(r) + \mathrm{l}(s)$.

2. $\left[\overline{2}^\sigma\right]^k(r)s \longrightarrow^* [r]^{2^k}(s)$

3. $[D]^k(r) \longrightarrow^* T_k(r)$

4. $T_k(I0) \longrightarrow^{2^k} T_k(0)$

Proof. All assertions are immediate by induction on k. □

Let $\overline{2}^{o(-1)}$ be D and $\overline{2}^{o(-2)}$ be $(I0)$. For natural numbers n and N we define a term A^n_N and compute one special reduction sequence of it using the last Lemma.

$$\begin{aligned} A^n_N &:= \left[\overline{2}^{o(n-1)}\right]^N (\overline{2}^{o(n-2)})\overline{2}^{o(n-3)} \ldots \overline{2}^{o(-2)} \\ &\overset{2.}{\longrightarrow}{}^* \left[\overline{2}^{o(n-2)}\right]^{2^N} (\overline{2}^{o(n-3)}) \ldots \overline{2}^{o(-2)} \overset{2.}{\longrightarrow}{}^* \ldots \\ &\overset{2.}{\longrightarrow}{}^* [D]^{2_n(N)}(I0) \overset{3.}{\longrightarrow}{}^* T_{2_n(N)}(I0) \overset{4.}{\longrightarrow}{}^{2_{n+1}(N)} T_{2_n(N)}(0). \end{aligned}$$

Then $\mathrm{d}(A^n_N) \geq 2_{n+1}(N)$, $\mathrm{g}(A^n_N) = \mathrm{g}(\overline{2}^{o(n-1)}) = n+1$ and $\mathrm{l}(A^n_N) = \mathrm{O}(N)$ independent of n, because $\mathrm{l}(A^{n+1}_N) = N \cdot \mathrm{l}(\overline{2}) + n \cdot \mathrm{l}(\overline{2}) + \mathrm{l}(D) + \mathrm{l}(I0)$ and $\mathrm{l}(A^0_N) = N \cdot \mathrm{l}(D) + \mathrm{l}(I0)$. Thus A^n_N has the desired properties.

Considering heights of terms we have to replace $\left[\overline{2}^{o(n-1)}\right]^N$ with height $\mathrm{O}(N)$ in the definition of A^n_N by a tree-like term with height $\mathrm{O}(N)$ which produces $\left[\overline{2}^{o(n-1)}\right]^{2^N}$. Let f be some variable of type $\sigma \to \sigma$. We define $b^\sigma_0 := f$ and $b^\sigma_{k+1} := \lambda x^\sigma . b^\sigma_k(b^\sigma_k x)$, then we have $b^\sigma_k x \longrightarrow^* [f]^{2^k}(x)$ as $\left[\overline{2}^\sigma\right]^k(f) \longrightarrow^* b^\sigma_k$, $\mathrm{g}(b^\sigma_k) = \mathrm{g}(f)$, $\mathrm{h}(b^\sigma_k) = 3 \cdot k$ and $\mathrm{h}(b^\sigma_k[f := r]) = \mathrm{h}(b^\sigma_k) + \mathrm{h}(r)$. With the abbreviations from above we define and compute

$$\begin{aligned} B^n_N &:= b^{o(n)}_N[f := \overline{2}^{o(n-1)}]\overline{2}^{o(n-2)} \ldots \overline{2}^{o(-2)} \\ &\longrightarrow^* \left[\overline{2}^{o(n-1)}\right]^{2^N} (\overline{2}^{o(n-2)})\overline{2}^{o(n-3)} \ldots \overline{2}^{o(-2)} \\ &\longrightarrow^* T_{2_{n+1}(N)}(I0) \longrightarrow^{2_{n+2}(N)} T_{2_{n+1}(N)}(0). \end{aligned}$$

Then $\mathrm{d}(B^n_N) \geq 2_{n+2}(N)$, $\mathrm{g}(B^n_N) = \mathrm{g}(\overline{2}^{o(n-1)}) = n+1$ and $\mathrm{h}(B^n_N) = \mathrm{O}(N)$, because $\mathrm{h}(B^{n+1}_N) = 3 \cdot N + \mathrm{h}(\overline{2}) + n + 2$ and $\mathrm{h}(B^0_N) = 3 \cdot N + \mathrm{h}(D) + 1$. Thus B^n_N has the desired properties.

Remark *The argument from §1 of [S82] applied to A^n_N reads as follows: Let S^n_N be the length of an arbitrary reduction sequence of A^n_N to its normal form $T_{2_n(N)}(0)$. As each reduction step at most squares the lengths of terms we obtain*

$$2_{n+1}(N) \leq \mathrm{l}(T_{2_n(N)}(0)) \leq \mathrm{l}(A^n_N)^{2^{S^n_N}} = (O(N))^{2^{S^n_N}} \leq 2^{2^{N+S^n_N}}$$

for N big enough. Hence $S^n_N \geq 2_{n-1}(N) - N$.

References

[B84] Barendregt, H.P.: *The Lambda Calculus.* Studies in Logic, Vo. 103, North-Holland (1984).

[G94] Goguen, H.: *A typed operational semantics for type theory.* PhD thesis, University of Edinburgh, 1994.

[L98] Loader, R.: *Notes on Simply Typed Lambda Calculus.* Technical Report ECS-LFCS-98-381, Edinburgh.

[R95] van Raamsdonk, F., Severi, P.: *On normalisation.* Computer Science Report CS-R9545 1995, Centrum for Wiskunde en Informatica.

[S82] Schwichtenberg, H.: *Complexity of normalization in the pure typed lambda - calculus.* In: The L.E.J.Brouwer Centenary Symposium, A.S. Troelstra and D. van Dalen (editors), North-Holland, 453–457 (1982).

[S91] Schwichtenberg, H.: *An upper bound for reduction sequences in the typed λ-calculus.* Arch. Math. Logic 30, 405–408 (1991).

[V87] de Vrijer, R.: *Exactly estimating functionals and strong normalization.* Nederl. Akad. Wetensch. Indag. Math. 49, 479–493 (1987)