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Separating fragments of bounded arithmetic

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Contents

1 Introduction **1**

1.1 Bounded arithmetic 1

1.2 Towards Dynamic Ordinals 3

1.3 Extended summary 5

2 Basic Definitions **13**

3 Pure Number Theory **17**

3.1 Preliminaries 17

3.2 The well-ordering proof in $\mathrm{I}\Sigma^0_n$ 19

4 Semi-formal Systems **25**

4.1 The infinitary language 25

4.2 The infinitary system 27

4.3 The Embedding of $\mathrm{I}\Sigma^0_n$ 29

5 Upper Bounds for $\mathcal{O}(\mathrm{I}\Sigma^0_n)$ **33**

5.1 $\mathcal{L}_\omega$-cut-elimination 33

5.2 The Boundedness Theorem 37

5.3 Applications: $\mathcal{O}(\mathrm{I}\Sigma^0_{n+1}) = \omega_{n+3}(0)$ and $\mathcal{O}(\mathrm{I}\Sigma^0_0) = \omega^2$ . . . 41

6 Notations for Exponentiation **45**

6.1 Exponential codes for natural numbers 45

6.2 Limited course-of-values recursion 48

6.3 $\mathcal{E}$, $\prec$, $\hat{+}$, $\mathrm{T}_\mathcal{E}$ are polytime 50

7 Bounded Predicative Arithmetic (BPA) **57**

7.1 The language 58

7.2 The theories 60

8 Bounded Arithmetic (BA) and BPA **67**
8.1 Fragments of BA 67
8.2 Comparing theories of BA 69
8.3 Comparing theories of BPA 72
8.4 Comparing BA with BPA 73

9 Well-ordering Proofs in BPA **75**
9.1 Formalization of wellfoundedness 75
9.2 What ᵖBASIC can prove 78
9.3 Proving foundation by induction 84

10 A Semi-formal System for BPA **87**
10.1 $\mathcal{L}^p_\omega$ and the semi-formal system BSF$^{\text{p}}$ 87
10.2 The embedding into BSF$^{\text{p}}$ 92
10.3 Extended cut-elimination I 97
10.4 Extended cut-elimination II 99

11 Predicative Boundedness Theorems (PBT) **101**
11.1 Preliminaries 101
11.2 Indiscernibles 102
11.3 Negative points and monotonicity 104
11.4 Proving PBT 106

12 Dynamic Ordinal Analysis (DOA) **111**
12.1 Dynamic Ordinals and separation 112
12.2 Computing Dynamic Ordinals 113
12.3 DOA in theories of BA 117

A Standard interpretations **121**

B $^{\text{p}}$BASIC **123**

C Proving indiscernibility **127**

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Chapter 1

Introduction

The present work represents the author's PhD-thesis at the *Mathematisch-Naturwissenschaftliche Fakultät* of the *Westfälische Wilhelms-Universität Münster*, which has been developed under supervision of Prof. W. Pohlers.

The aim of this work is to investigate proof-theoretically formal theories of bounded arithmetic. For this purpose the subsystems $\mathrm{I}\Sigma^0_n$ of first order arithmetic and subsystems of bounded predicative arithmetic will be investigated, too.

1.1 Bounded arithmetic

"Bounded arithmetic theories are subtheories of first order arithmetic. They attempt to formalize reasoning about finite structures"[1]. In [6] S. Buss introduced the theories S^n_2, T^n_2, U^1_2, V^1_2 of bounded arithmetic which correspond to the computational classes in the *polynomial time hierarchy* **PH**, **PSPACE** and **EXPTIME**. The classes **P** of languages computable in polynomial time on deterministic Turing-machines and **NP** of languages computable in polynomial time on non-deterministic Turing-machines are levels of **PH**.

It is a common assumption that the separation problems of bounded arithmetic theories are essentially the same as the separation problems of computational classes (including **P** vs. **NP**), although the only known result in this relation is the following result in [16]:

$$\mathrm{T}^n_2 = \mathrm{S}^{n+1}_2 \implies \Sigma^{\mathrm{p}}_{n+2} = \Pi^{\mathrm{p}}_{n+2}.$$

[1]See [9] p. 2.

Thus, the collapse of S_2 implies the collapse of **PH**.[2] Therefore, the separation problems of bounded arithmetic theories are among the major unsolved problems of the present time.

In the case of relativized computational classes things are quite different. It has been shown in [1] that there are oracles A and B such that $\mathbf{P}^A = \mathbf{NP}^A$ and $\mathbf{P}^B \neq \mathbf{NP}^B$. In [25] and also in [13] it has been shown that there is an oracle A such that $\mathbf{PH}^A$ (i.e., the polynomial time hierarchy with an oracle A) does not collapse.

Corresponding results for *bounded arithmetic theories* are proved by using these results. The set $\Sigma^b_\infty(\mathcal{X})$ of *bounded formulas* of the language of bounded arithmetic with set parameters $X_0, X_1, \ldots$ is stratified into levels $\Sigma^b_0(\mathcal{X}) \subset \Sigma^b_1(\mathcal{X}) \subset \ldots$ similar as the arithmetical formulas are stratified into levels $\Sigma^0_0(\mathcal{X}) \subset \Sigma^0_1(\mathcal{X}) \subset \ldots$. More precisely, $\Sigma^b_0(\mathcal{X})$ is the set of bounded formulas where all quantifiers are sharply bounded quantifiers (i.e., they are bounded by a term of the form $|t|$, where $|n| = \lceil \log_2(n+1) \rceil$). In addition to this $\Sigma^b_{i+1}(\mathcal{X})$ is the set of bounded formulas with i alternations of bounded quantifiers, which start with an existential one and do not count the sharply bounded ones. The *prenex* (or *strict*) versions of $\Sigma^b_i(\mathcal{X})$ (where the closure under sharply bounded quantifiers is omitted) are denoted by $s\Sigma^b_i(\mathcal{X})$. The sets of bounded formulas without set variables will be denoted omitting "$(\mathcal{X})$".

Let $|y|_0 := y$ and $|y|_{m+1} := |(|y|_m)|$. The theories $\Sigma^b_n(\mathcal{X})$-L^mInd are axiomatized by a finite set of defining axioms for the non-logical symbols and by the induction schema which consists of all formulas of the form

$$F(0) \land \forall x{<}|t|_m\, (F(x) \to F(x+1)) \to F(|t|_m)$$

with $F \in \Sigma^b_n(\mathcal{X})$ and t being a term. As exponentiation $\lambda n.2^n$ is not a function which can be proved to be total in bounded arithmetic, this induction schema seems to become weaker if m increases. The theories with small numbers m have special names:

$$\begin{aligned} sR^n_2(\mathcal{X}) &:= s\Sigma^b_n(\mathcal{X})\text{-}L^2\text{Ind} \\ R^n_2(\mathcal{X}) &:= \Sigma^b_n(\mathcal{X})\text{-}L^2\text{Ind} \\ S^n_2(\mathcal{X}) &:= \Sigma^b_n(\mathcal{X})\text{-}L^1\text{Ind} \\ T^n_2(\mathcal{X}) &:= \Sigma^b_n(\mathcal{X})\text{-}L^0\text{Ind}. \end{aligned}$$

---

[2] Cf. [24].

Theories without set variables are denoted by sR_2^n, S_2^n ... ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{T}_2^n$ ... .

It holds that:

- $\mathrm{S}_2^n(\mathcal{X}) \subset \mathrm{T}_2^n(\mathcal{X}) \subset \mathrm{S}_2^{n+1}(\mathcal{X})$ ([6]),
- $\mathrm{S}_2^{n+1}(\mathcal{X})$ is $\forall\Sigma_{n+1}^{\mathrm{b}}(\mathcal{X})$-conservative over $\mathrm{T}_2^n(\mathcal{X})$ ([7]), although
- $\mathrm{S}_2^{n+1}(\mathcal{X}) \neq \mathrm{T}_2^n(\mathcal{X})$ ([16]).
- $\mathrm{T}_2^n(\mathcal{X}) \neq \mathrm{S}_2^n(\mathcal{X})$ ([14]), thus
- $\mathrm{T}_2^n(\mathcal{X})$ is not $\forall\Sigma_n^{\mathrm{b}}(\mathcal{X})$-conservative over $\mathrm{S}_2^n(\mathcal{X})$ ([8])

Here $\forall\Sigma_i^{\mathrm{b}}(\mathcal{X})$ is the set of first order universal closures of formulas from $\Sigma_i^{\mathrm{b}}(\mathcal{X})$.

All these separation results are proved by showing that a certain principle $\Phi(X)$ is not witnessed in polynomial time by a TURING-machine with an oracle from Σ_i^{p} (fixed i for all X). In this thesis we develop a new concept, which allows us to uniformly prove separation results for bounded arithmetic theories.

1.2 Towards Dynamic Ordinals

GENTZEN's consistency proof for pure number theory[3] was the starting point of *ordinal analysis*. Ordinal analysis assigns a characteristic value to a formal system, its *proof-theoretical ordinal*. The proof-theoretical ordinal $\mathcal{O}(S)$ of a formal system S with an associated concept of formal derivability $\vdash$ is defined as the supremum of the ordertypes $|| \prec ||$ of primitive recursive definable well-orderings $\prec$ whose wellfoundedness can be recognized in S:

$$\mathcal{O}(S) = \sup\{|| \prec || : \prec \text{ is a primitive recursive definable well-ordering and } S \vdash \mathit{Fund}(\prec, X)\}.$$

The formula $\mathit{Fund}(\prec, X)$ describes that if $\prec$ is progressive on X then X is total:

$$\mathit{Fund}(\prec, X) :\equiv \forall x\,(\forall y\,(y \prec x \rightarrow y \in X) \rightarrow x \in X) \rightarrow \forall x\,(x \in X).$$

[3] See [11].

Thus $\forall X\, \mathit{Fund}(\prec, X)$ expresses the wellfoundedness of $\prec$.

Different proof-theoretical ordinals imply a separation of the underlying systems: if $\mathcal{O}(S_1) \not\subseteq \mathcal{O}(S_2)$ then there is some well-ordering $\prec$ such that $S_1 \vdash \mathit{Fund}(\prec, X)$ but $S_2 \not\vdash \mathit{Fund}(\prec, X)$.

$\mathrm{I}\Sigma^0_1$ is the theory of first order arithmetic with induction restricted to $\Sigma^0_1(\mathcal{X})$-*formulas*, i.e., formulas of the form: one unbounded existential quantifier followed by a bounded formula. For arithmetic theories which are strong enough (i.e., which are extensions of $\mathrm{I}\Sigma^0_1$) the proof-theoretical ordinal is a *good measurement* in the sense that the different theories under consideration receive different proof-theoretical ordinals.

For subsystems of $\mathrm{I}\Sigma^0_1$ the proof-theoretical ordinal does not yield a good measurement. R. Sommer has shown in [20]:

$$\mathrm{I}\Sigma^0_0 \vdash \mathit{Fund}(\omega \cdot k, F) \text{ for all } k < \omega, F \in \Sigma^0_0(\mathcal{X})$$

and

$$\mathrm{I}\Sigma^0_0 + \mathit{Fund}(\omega^2, \Sigma^0_0(\mathcal{X})) = \mathrm{I}\Sigma^0_1.$$

For bounded arithmetic theories he remarks in [21]:

$$\mathrm{T}^1_2(\mathcal{X}) \vdash \mathit{Fund}(\omega \cdot k, F) \text{ for all } k < \omega, F \in \Delta^{\mathrm{b}}_0(\mathcal{X})$$

and

$$\mathrm{S}^1_2(\mathcal{X}) + \mathit{Fund}(\omega^2, \Delta^{\mathrm{b}}_0(\mathcal{X})) = \mathrm{I}\Sigma^0_1.$$

Therefore we obtain

$$\mathcal{O}(T) = \omega^2$$

for theories T which are stronger than $\mathrm{T}^1_2(\mathcal{X})$ but weaker than $\mathrm{I}\Sigma^0_1$.

Let T be a subsystem of first order arithmetic. Let $\mathcal{S}$ be a suitable ordinal notation system for T, and let $\Phi : \mathcal{S} \to \mathrm{On}$ be the associated evaluation function. Ordinal analysis is statically in the sense that it determines firm natural numbers $n \in \mathcal{S}$ coding ordinals, such that T proves the wellfoundedness of $\Phi(n)$. As illustrated above this yields no information for weak theories – we always obtain the same proof-theoretical ordinal. This deficiency can be overcome by *Dynamic Ordinals*. We consider functions $F : \omega \to \mathcal{S}$ enumerating natural numbers which code ordinals such that T proves the wellfoundedness of $\Phi(F(n))$ *uniform* in n. Now we have the chance that considering the growth rates of such functions yields a good measurement for weak theories. Thereby we do not think of the growth of the values of F

according to the canonical ordering of the natural numbers – F can have (and, of course, will have in the analysis of bounded arithmetic theories) polynomial growth rate. Here we mean growth of the values of F according to the canonical ordering of the coded ordinals.

We define this formally. Let ${}^{\omega}(\Phi[\mathcal{S}])$ be the set of all functions $f : \omega \to \Phi[\mathcal{S}]$. For $f, g \in {}^{\omega}(\Phi[\mathcal{S}])$ define $f \leq g$ iff f *is majorized by* g, i.e., $\forall n\,(f(n) \leq g(n))$. For $\mathcal{F} \subset {}^{\omega}(\Phi[\mathcal{S}])$ let the $\leq$-hull of $\mathcal{F}$ be

$$\mathcal{H}(\mathcal{F}) := \{f \in {}^{\omega}(\Phi[\mathcal{S}]) \,:\, \exists g{\in}\mathcal{F}\,(f \leq g)\}.$$

Then we define the *Dynamic Ordinal* of T, $\mathcal{DO}(T)$, by

$$\begin{aligned}\mathcal{DO}(T) := \mathcal{H}\{\lambda n.\Phi(F(n)) \mid F : \omega \to \mathcal{S} &\text{ is a provable recursive}\\ \text{function of } T \text{ and } T \vdash \forall x\, \mathit{Fund}(F(x), X)\}.\end{aligned}$$

For theories stronger than or equal to $\mathrm{I}\Sigma^0_1$ Dynamic Ordinals yield no additional information when compared with proof-theoretical ordinals. E.g., let $\mathcal{S}$ be the common ordinal notation system for ε_0, then $\mathcal{O}(\mathrm{I}\Sigma^0_n) < \mathcal{O}(\mathrm{I}\Sigma^0_{n+1})$. Thus all functions in $\mathcal{DO}(\mathrm{I}\Sigma^0_n)$ can be majorized by the constant function $\lambda n.$ "code of $\mathcal{O}(\mathrm{I}\Sigma^0_n) + 1$" which is in $\mathcal{DO}(\mathrm{I}\Sigma^0_{n+1})$.

Different Dynamic Ordinals imply a separation of the assigned theories: if there is an $f \in \mathcal{DO}(T_2) \setminus \mathcal{DO}(T_1)$ then by definition there is a function $F : \omega \to \mathcal{S}$ which is provable recursive in T_2 such that

$$T_2 \vdash \forall x\, \mathit{Fund}(F(x), X)$$

and $f \leq \big(\lambda n.\Phi(F(n))\big) =: g$. Now $f \notin \mathcal{DO}(T_1)$ yields $g \notin \mathcal{DO}(T_1)$, thus F is not provable recursive in T_1 or

$$T_1 \not\vdash \mathit{Fund}(F(x), X).$$

We will see that Dynamic Ordinals give us good measurements for bounded arithmetic theories.

1.3 Extended summary

The methods of ordinal analysis for first order arithmetic and its subsystems $\mathrm{I}\Sigma^0_n$ form a basis for the investigations of the bounded arithmetic theories $\mathrm{S}^n_2(\mathcal{X})$, $\mathrm{T}^n_2(\mathcal{X})$, etc. These methods are composed of

- carrying through the well-ordering proof in the formal system. This yields a lower bound for the (dynamic) proof-theoretical ordinal.

- formulating a semi-formal system and proving
  - cut-elimination
  - that formal derivations can be embedded into the semi-formal system
  - a so called Boundedness Principle for the semi-formal system: a (almost) cut-free semi-formal derivation of the well-foundedness of a well-ordering needs at least α steps, where α is the ordertype of the well-ordering.

  This yields an upper bound for the (dynamic) proof-theoretical ordinal.

In the first part, from Chapter 3 to Chapter 5, we work out these methods for the systems $\mathrm{I}\Sigma^0_n$ and obtain the well-known results

$$\begin{aligned}\mathcal{O}(\mathrm{I}\Sigma^0_0) &= \omega^2 \\ \mathcal{O}(\mathrm{I}\Sigma^0_{n+1}) &= \omega_{n+3}(0),\end{aligned}$$

where $\omega_0(\alpha) = \alpha$ and $\omega_{i+1}(\alpha) = \omega^{\omega_i(\alpha)}$. In this part two main results are new. The first one is an $\mathcal{L}_\omega$*-cut-elimination* which shows that a cut with a propositional formula can be substituted by as many cuts of atomic formulas as the formula contains atoms. This avoids exponential growth of derivation lengths, a consequence of the usual cut-elimination procedure. The second result is a *sharpened version of the Boundedness Theorem* which goes back to GENTZEN. The original version, of which a proof can be found in [17], states

$$\vdash^{\alpha}_{1} \mathit{Fund}(\prec, X) \implies \mathcal{O}(\prec) \le 2^{\alpha}.$$

We use a new idea to prove

$$\vdash^{\alpha}_{1} \mathit{Fund}(\prec, X) \implies \mathcal{O}(\prec) \le \alpha.$$

Again this avoids additional exponential growth.

Ordinal analysis always uses cut-elimination which involves exponential growth of derivation lengths. Therefore, if we try to transfer methods from ordinal analysis to bounded arithmetic we have to find a way of dealing with the exponential function even in bounded arithmetic theories, although these theories cannot prove the totality of the usual exponential function.

In the ordinal analysis of $I\Sigma_n^0$ similar problems occur when we try to speak about ordinals and the function $\lambda\alpha.\omega^\alpha$. The solution there is to code ordinals by natural numbers. Replacing ω by 2 transfers this idea to the situation of bounded arithmetic. Therefore, we obtain the following correspondences:

first order arithmetic	bounded arithmetic
"$\lambda\alpha.\omega^\alpha$"	"$\lambda n.2^n$"
For $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} < \varepsilon_0$ with $\alpha_n \leq \ldots \leq \alpha_1$ let	For $\alpha = 2^{\alpha_1} + \ldots + 2^{\alpha_n} < \omega$ with $\alpha_n < \ldots < \alpha_1$ let
$\hat{\alpha} := \langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle \in \mathcal{D} \subset \omega.$	$\hat{\alpha} := \langle \hat{\alpha}_1, \ldots, \hat{\alpha}_n \rangle \in \mathcal{E} \subset \omega.$
Thus, we obtain $\lambda\alpha.\omega^\alpha$ as a provably recursive function on the notations $\mathcal{D}$:	Thus, we obtain $\lambda\alpha.2^\alpha$ as a provably recursive function on the notations $\mathcal{E}$:
$\lambda\alpha.\hat{\omega}^\alpha := \langle\alpha\rangle$	$\lambda\alpha.\hat{2}^\alpha := \langle\alpha\rangle$

The coding-functions $\langle\ldots\rangle$ are the familiar GÖDEL numbers for sequences[4], which are polynomial time computable. In Chapter 6 we will show that the exponential notations and several basic operations on them are polynomial time computable functions.

In the last part, from Chapter 7 to Chapter 12, we apply the methods described above to bounded arithmetic theories in order to obtain a good measurement of those theories. To this end we have to find, beside other things, a bounded formula which describes the wellfoundedness of the ordering $\prec$ on the exponential notations up to some exponential notation α. $\prec$ is given according to the ordering of the coded values, i.e., $\prec$ satisfies $\hat{m} \prec \hat{n} \iff m < n$. It would suffice to find a value a which bounds all exponential notations below α:

$$\forall\beta \prec \alpha\ (\beta \leq a).$$

[4]Cf. [6] p. 7.

Then all unbounded quantifiers in $Fund(\alpha, X)$ can be bounded by a. This yields the desired formula $Fund(a, \alpha, X)$.

Let $\Phi_{\mathcal{E}}$ be defined by $\Phi_{\mathcal{E}}(\alpha(x)) = n \iff \hat{n} = \alpha(x)$. A function $f(x)$ is *polynomially bounded* if there is a polynomial $p(x)$ such that $|f(x)| \leq p(|x|)$. If α is given by a function $\alpha(x)$ of x and the value $\Phi_{\mathcal{E}}(\alpha(x))$ denoted by $\alpha(x)$ is close enough to x, i.e., it is *polynomially bounded*, then we obtain the desired value a as a function of x in bounded arithmetic, i.e., we discover it as a term in x of the language of bounded arithmetic. Therefore, the methods described above can be applied to some bounded arithmetic theories and they produce a good measurement of them.

On the other hand if $\Phi_{\mathcal{E}}(\alpha(x))$ is not close enough to x, which means that eventually $\Phi_{\mathcal{E}}(\alpha(x)) \geq 2^x$, the only expedient is to *assume* the existence of such a value a. This value is not allowed to bound the length of an induction – otherwise this would influence the Dynamic Ordinal in a way that a in general cannot bound all exponential notations below this Dynamic Ordinal. Thus, from the point of view of induction, a has to be *impredicative*. The linguistic frame in which this takes place will be called *bounded predicative arithmetic*. It leads to conservative extensions of bounded arithmetic theories ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X})$ and ${}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X})$.

Models of bounded predicative arithmetic theories distinguish between two kinds of individuals, the *predicative* ones (from I_p) and the *impredicative* ones (from I). It holds:

- I_p is a subset of I.
- I_p is closed under some polynomial time computable functions, e.g. $+$, $\cdot$ and the "smash"-function $x \# y := 2^{|x|\cdot|y|}$, and it admits weak induction principles depending on the underlying theory.
- On I only graphs of some polynomial time computable functions are given.

All this will be introduced in Chapter 7. In Chapter 8 we summarize the relationships between previously defined bounded arithmetic theories, transfer them to bounded predicative arithmetic theories and finally show that the latter theories are conservative extensions of the corresponding former ones.

A specific bounded formula $Big(a, b, \alpha)$ solves the previously explained difficulties which lead to bounded predicative arithmetic. This formula expresses that all exponential notations below α are bounded by a

$$\forall \beta \prec \alpha \; (\beta \leq a),$$

and that the graphs $\mathcal{G}_f$ which are under consideration in the language of bounded predicative arithmetic define total functions with values below b applied to arguments below a

$$\forall \vec{c} {\leq} a \, \exists d {\leq} b \, \mathcal{G}_f(\vec{c}, d).$$

This leads to the formula

$$Big(a, b, \alpha) \to Fund(a, \alpha, X).$$

Following a suggestion of Jan Krajíček in Prague in August 1996 we will abbreviate this formula by $BigFun(a, b, \alpha, X)$. Why not, as it yields so many exciting separation results.

For bounded (predicative) arithmetic theories T we define the *Dynamic Ordinal of* T, $\mathcal{DO}(T)$, by

$$\begin{aligned} \mathcal{DO}(T) \;:=\; \mathcal{H}\Big(\{\lambda n.\Phi_{\mathcal{E}}(t(n)) \mid & \; t(x) \text{ is a term} \\ & \text{defining a function } t(.) : \omega \to \mathcal{E} \\ & \text{with } T \vdash \forall x \, BigFun(a, b, t(x), X)\}\Big). \end{aligned}$$

In Chapters 9 to 11 we transfer the techniques developed in Chapters 3 to 5 for the theories $\mathrm{I}\Sigma^0_n$ to bounded predicative arithmetic. Furthermore, we show that nearly the same works if we replace the set variable X with the set $X(d) = \{i : \mathrm{Bit}(i, d)\}$ coded by the impredicative value d in $BigFun$, where $\mathrm{Bit}(i, d)$ is true iff the i-th bit in the binary expansion of d is 1. This replacement requires the existence of *indiscernibles*: to a given (finite) set Π of formulas and $l \in \omega$ there exists a set $I \subset \omega$ of indiscernibles such that

$$\forall M \subset \{0, \ldots, l\} \; \exists m {\in} I \, (m \text{ codes } M \text{ below } l),$$

at which a number m codes a set M below l iff $\forall i {\leq} l \, (i {\in} M \leftrightarrow \mathrm{Bit}(i, m))$, and

$$\exists m {\in} I \, (\mathbb{N} \models A_d[m]) \iff \forall m {\in} I \, (\mathbb{N} \models A_d[m])$$

for all atomic formulas $A \in \Pi$ other than $\mathrm{Bit}(\cdot, d)$ or $\mathrm{Bit}^{\mathrm{c}}(\cdot, d)$, where $\mathrm{Bit}^{\mathrm{c}}(\cdot, d)$ is the complement of $\mathrm{Bit}(\cdot, d)$. The indiscernibles are essential for the monotonicity of formulas $F \in \Pi$ in which $\mathrm{Bit}^{\mathrm{c}}(\cdot, d)$ does not occur (and again this kind of monotonicity is essential for the proof of the Predicative Boundedness Theorem):

$$m, n \in I,\ m \subset n \ \&\ \mathbb{N} \vDash F_d[m] \implies \mathbb{N} \vDash F_d[n],$$

at which $\forall i\,(\mathrm{Bit}(i, m) \to \mathrm{Bit}(i, n))$.
Results: Let $n + 1 \geq m \geq 1$, then

$$\begin{aligned}
\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-L}^m\mathrm{Ind}) &= \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-L}^m\mathrm{Ind}) \\
&= \mathcal{H}(\{\lambda i.2_n(p(|i|_m)) \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+1}_2) &= \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+1}_2(\mathcal{X})) \\
&= \mathcal{H}(\{\lambda i.2_n(p(|i|)) \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{n+2}_2) &= \mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{n+2}_2(\mathcal{X})) \\
&= \mathcal{H}(\{\lambda i.2_{n+1}(p(||i||)) \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2) &= \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X})) \\
&= \mathcal{H}(\{\lambda i.2_{n+1}(p(|i|)) \,:\, p \text{ a polynomial}\}).
\end{aligned}$$

Furthermore, for $n \geq 0$ the results are

$$\begin{aligned}
\mathcal{DO}(\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-L}^{n+1}\mathrm{Ind}) &= \mathcal{H}(\{\lambda i.2_n(p(|i|_{n+1})) \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{S}^1_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.p(|i|) \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{sR}^2_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.2^{p(||i||)} \,:\, p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{T}^1_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.2^{p(|i|)} \,:\, p \text{ a polynomial}\}) \\
&= \mathcal{DO}(\mathrm{S}^2_2(\mathcal{X})).
\end{aligned}$$

For theories T_1, T_2 let $T_1 \subseteq T_2$ iff T_1 is included in T_2, which means that for all formulas F if $T_1 \vdash F$ then $T_2 \vdash F$. Let $T_1 \subsetneq T_2$ iff T_2 is a proper extension of T_1, i.e., $T_1 \subseteq T_2$ and $T_1 \not\supseteq T_2$. Let $n \geq 0$ and $m \geq 1$. The results imply the following relations between bounded predicative theories:

$$\begin{array}{ccc}
 & {}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X}) & \\
\not\nwarrow & & \not\nearrow \\
{}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}(\mathcal{X})\text{-L}^m\mathrm{Ind} & \subsetneq & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}(\mathcal{X})\text{-L}^{m+1}\mathrm{Ind}.
\end{array}$$

Hence

$$\begin{array}{ccccc}
{}^{\mathrm{p}}\mathrm{S}_2^{n+1}(\mathcal{X}) & \subsetneq & {}^{\mathrm{p}}\mathrm{T}_2^{n+1}(\mathcal{X}) & & \\
& \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & \\
& & {}^{\mathrm{p}}\mathrm{R}_2^{n+2}(\mathcal{X}) & \subsetneq & {}^{\mathrm{p}}\mathrm{S}_2^{n+2}(\mathcal{X}).
\end{array}$$

For bounded predicative arithmetic theories without set variables we also show:

$$\begin{array}{ccccc}
& & {}^{\mathrm{p}}\mathrm{T}_2^{n+1} & & \\
& \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\not\subseteq$} & \\
{}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}\text{-}\mathrm{L}^m\mathrm{Ind} & & \subsetneq & & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-}\mathrm{L}^{m+1}\mathrm{Ind}.
\end{array}$$

Hence

$$\begin{array}{ccccc}
{}^{\mathrm{p}}\mathrm{S}_2^{n+1} & \subsetneq & {}^{\mathrm{p}}\mathrm{T}_2^{n+1} & & \\
& \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & \\
& & {}^{\mathrm{p}}\mathrm{R}_2^{n+2} & \subsetneq & {}^{\mathrm{p}}\mathrm{S}_2^{n+2}.
\end{array}$$

For small bounded arithmetic theories we obtain:

$$\begin{array}{ccccc}
& & \mathrm{T}_2^1(\mathcal{X}) & & \\
& \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\not\subseteq$} & \\
\mathrm{s}\Sigma^{\mathrm{b}}_m(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} & & \subsetneq & & \mathrm{s}\Sigma^{\mathrm{b}}_{m+1}(\mathcal{X})\text{-}\mathrm{L}^{m+1}\mathrm{Ind}.
\end{array}$$

Hence

$$\begin{array}{ccccc}
\mathrm{S}_2^1(\mathcal{X}) & \subsetneq & \mathrm{T}_2^1(\mathcal{X}) & & \\
& \rotatebox{45}{$\subsetneq$} & & \rotatebox{-45}{$\supsetneq$} & \\
& & \mathrm{sR}_2^2(\mathcal{X}) & \subsetneq & \mathrm{S}_2^2(\mathcal{X}).
\end{array}$$

Chapter 2

Basic Definitions

We fix:

- The set of the natural numbers is always identified with the ordinal $\omega = \{0, 1, 2, \ldots\}$. Let $\mathfrak{P}(\omega)$ be the power set of ω, i.e., $\mathfrak{P}(\omega) := \{S : S \subset \omega\}$.
- We denote the first uncountable ordinal by Ω.
- Let $\omega_0(\alpha) := \alpha$ and $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$. Let $2_0(\alpha) := \alpha$ and $2_{n+1}(\alpha) := 2^{2_n(\alpha)}$.
- Sometimes we will use a dyadic notation of the natural numbers: let $i_j \in \{0, 1\}$ for $j \leq k$, then we define

  $$(i_k \ldots i_0)_2 := \sum_{j=0}^{k} i_j \cdot 2^j.$$

  We shortly write $(si_k \ldots i_0)_2$ for $s \cdot 2^{k+1} + (i_k \ldots i_0)_2$ if $s > 1$.
- Let $S_\varphi(t)$ be the expression obtained from the string S by replacing all occurrences of φ by t.
- Let $A(\cdot) :\equiv \{\varphi : A(\varphi)\}$.

In the following we introduce some basic polynomial time computable functions which will be of interest in the further development of this thesis. From now on we abbreviate "polynomial time computable" by "*polytime*".

- S, $+$, $\cdot$ are the usual successor, addition and multiplication functions.

- S_0 and S_1 are the binary successor functions given by $\lambda n.\, S_i(n) = 2 \cdot n + i$ with $i \in \{0, 1\}$.
- The binary length function is given by $\lambda n.|n| = \lceil \log_2(n+1) \rceil$, where we set $\lceil r \rceil$ for real numbers r as the least integer z which is bigger than or equal to r. For sequences $n_1, \ldots, n_k$ we shortly write $|n_1, \ldots, n_k|$ instead of $|n_1|, \ldots, |n_k|$.
- The shift right function $\lambda n.\lfloor \frac{1}{2} n \rfloor$ assigns to each natural number n the biggest natural number which is less than or equal to $\frac{n}{2}$.
- The smash function is given by $\lambda mn.m \# n = 2^{|m| \cdot |n|}$.
- The arithmetical subtraction function is defined by

$$\lambda mn.m \dot{-} n = \begin{cases} m - n & if\ m - n \geq 0 \\ 0 & otherwise. \end{cases}$$

- The functions $\lambda mn.\, \mathrm{MSP}(m, n)$ and $\lambda mn.\, \mathrm{LSP}(m, n)$ compute the more significant part and the less significant part of a natural number m. They are uniquely defined by the conditions

$$m = \mathrm{MSP}(m, n) \cdot 2^n + \mathrm{LSP}(m, n) \quad and \quad \mathrm{LSP}(m, n) < 2^n.$$

- $<$ and $\leq$ are the usual "less than" and "less than or equal" relations.
- The predicate $\mathrm{Bit}(m, n)$ is true iff the m-th bit in the binary expansion of n is 1.

All *polytime functions* are generated from the basic functions defined above using composition and the following rules of limited recursion on notations[1] or of limited recursion[2].

The function f is defined from functions g, h_0, h_1 and k by *limited recursion on notation* if

$$\begin{aligned} f(\vec{x}, 0) &= g(\vec{x}) \\ f(\vec{x}, S_i(y)) &= h_i(\vec{x}, y, f(\vec{x}, y)) \quad (i = 0, 1; \quad i \neq 0 \text{ if } y = 0) \end{aligned}$$

provided that $f(\vec{x}, y) \leq k(\vec{x}, y)$ for all $\vec{x}, y$.

[1] Cf. [10] p. 28.

[2] Cf. [6] p. 8.

See Rose [18] for a proof that this rule again defines polytime functions.

The function f is defined from functions g, h and polynomials p and q by *limited recursion* if the following holds:
Let the function τ be defined as

$$\begin{aligned} \tau(\vec{x}, 0) &= g(\vec{x}) \\ \tau(\vec{x}, \mathrm{S}(y)) &= h(\vec{x}, y, \tau(\vec{x}, y)). \end{aligned}$$

Then let

$$f(\vec{x}) = \tau(\vec{x}, p(|\vec{x}|))$$

provided that $|\tau(\vec{x}, y)| \leq q(|\vec{x}|)$ for all $\vec{x}$ and $y \leq p(|\vec{x}|))$.

See Buss [6] for a proof that this rule again defines polytime functions.

A *monotone polynomial* is a polynomial containing only positive coefficients. All polytime functions have *polynomial growth rate*, that means: given a polytime function f there is some monotone polynomial q_f such that

$$|f(\vec{n})| \leq q_f(|\vec{n}|)$$

for all $\vec{n} \in \omega$. It is well-known that for each monotone polynomial q there exists a polytime function f_q such that

$$q(|\vec{n}|) \leq |f_q(\vec{n})|$$

for all $\vec{n} \in \omega$. This function f_q can be defined as a term from 0, $\vec{n}$ and $\#$.

Chapter 3

Pure Number Theory

In the following three chapters we do the *ordinal analysis* for the subsystems $\mathrm{I}\Sigma_n^0$ of pure number theory Z_1. I.e., we compute the proof-theoretical ordinal $\mathcal{O}(\mathrm{I}\Sigma_n^0)$ which is the supremum of the ordertypes of all primitive recursive definable order relations whose wellfoundedness is provable in $\mathrm{I}\Sigma_n^0$. *Pure number theory* Z_1 is an extension of PEANO arithmetic by definitions. In Z_1 there are symbols for all primitive recursive functions and set variables.

3.1 Preliminaries

Let us fix a first order language $\mathcal{L}_{Z_1}$ with equality in which the fragments of Z_1 which are under consideration can be axiomatized. We adopt the definition from POHLERS [17] with the change that we use a *language in* TAIT*-style*, i.e., without a negation symbol – negation will be defined syntactically.

The *logical symbols* of $\mathcal{L}_{Z_1}$ are: countably many number variables $x_0, x_1, \ldots$, countably many set variables $X_0, X_1, \ldots$, the sentential connectives $\wedge, \vee$, the quantifiers $\forall, \exists$, the equality symbols $=, \neq$ and the membership relation symbols $\in, \notin$.

The *nonlogical symbols* of $\mathcal{L}_{Z_1}$ are: a constant $\underline{n}$ for each natural number n, an n-ary function symbol $\underline{f}$ for each n-ary primitive recursive function f and brackets as auxiliary symbols. We consider $<, \not<, \leq$ and $\not\leq$ as defined symbols. There is no negation symbol in $\mathcal{L}_{Z_1}$ but we can define a syntactic operation $\neg : \mathcal{L}_{Z_1} \to \mathcal{L}_{Z_1}$ which has the meaning of negation according to the DE MORGAN laws, see [17] p. 23. We abbreviate $\neg F \vee G$ by $F \to G$ and $(F \to G) \wedge (G \to F)$ by $F \leftrightarrow G$.

The set of *terms of* $\mathcal{L}_{Z_1}$ is the smallest set which contains the number variables and constants and is closed under the function symbols of $\mathcal{L}_{Z_1}$. $\mathcal{L}_{Z_1}$-terms which contain no free number variables are called *ground terms*. Let $t^{\mathbb{N}}$ be the evaluation of a ground term t according to the standard interpretation of the constants and the function symbols. *Prime formulas* or *atomic formulas* are of the form $s = t, s \neq t, s \in X$ or $s \notin X$ where s, t are terms and X is a set variable of $\mathcal{L}_{Z_1}$. We obtain all *first order* $\mathcal{L}_{Z_1}$*-formulas* from atomic formulas by closing under $\vee, \wedge, \exists, \forall$. We abbreviate $\forall x\,(x \leq t \rightarrow A)$ and $\exists x\,(x \leq t \wedge A)$ by $\forall x{\leq}t\,A$ resp. $\exists x{\leq}t\,A$. These quantifiers are called *bounded quantifiers*, quantifiers not of this form are called *unbounded quantifiers*. $\mathcal{L}_{Z_1}$-formulas which do not contain free set variables are called *arithmetical*. $\mathcal{L}_{Z_1}$-formulas which do not contain free number variables are called Π^1_1*-sentences of* $\mathcal{L}_{Z_1}$. Arithmetical $\mathcal{L}_{Z_1}$-formulas which do not contain free number variables are called *sentences of* $\mathcal{L}_{Z_1}$.

Let F be a Π^1_1-sentences containing no variable not occuring in $Y_1, \ldots, Y_k$. Let $M_1, \ldots, M_k \in \mathfrak{P}(\omega)$, then $\mathbb{N} \vDash F_{Y_1,\ldots,Y_k}[M_1, \ldots, M_k]$ is defined as usual[1]. Let $\mathbb{N} \vDash F$ iff $\mathbb{N} \vDash F_{Y_1,\ldots,Y_k}[M_1, \ldots, M_k]$ for all $M_1, \ldots, M_k \in \mathfrak{P}(\omega)$.

In order to axiomatize the fragments of Z_1 in question we first define some special sets of $\mathcal{L}_{Z_1}$-formulas: $\Sigma^0_0 = \Pi^0_0 = \Delta^0_0$ is the smallest set of $\mathcal{L}_{Z_1}$-formulas which contains all atomic formulas and is closed under sentential connectives and bounded quantification. Σ^0_{n+1} is the set of $\mathcal{L}_{Z_1}$-formulas of the form $\exists x\,A$ with $A \in \Pi^0_n$. Π^0_{n+1} is the set of $\mathcal{L}_{Z_1}$-formulas of the form $\forall x\,A$ with $A \in \Sigma^0_n$.

Let $\mathrm{BASIC}_{\mathrm{Z}_1}$ be some convenient set of $\mathcal{L}_{Z_1}$-sentences which axiomatizes the nonlogical symbols of $\mathcal{L}_{Z_1}$[2], i.e., $\mathrm{BASIC}_{\mathrm{Z}_1}$ consists of the defining equations for the constants and the recursion equations for the function symbols. We define *induction axioms* depending on sets of $\mathcal{L}_{Z_1}$-formulas Φ: let (Φ-IND) be the set consisting of the universal closure of formulas

$$A(\underline{0}) \wedge \forall x\,(A(x) \rightarrow A(\mathrm{S}\,x)) \rightarrow \forall x\,A(x)$$

[1]See [17] pp. 14 for a definition.

[2]In [17] pp. 18 we can find a suitable axiomatization.

with $A(x) \in \Phi$. We consider the axiom systems

$$\begin{aligned} \mathrm{Z}_1 &= \mathrm{BASIC}_{\mathrm{Z}_1} \cup (\mathcal{L}_{Z_1}\text{-IND}) \\ \mathrm{I}\Sigma^0_n &= \mathrm{BASIC}_{\mathrm{Z}_1} \cup (\Sigma^0_n\text{-IND}). \end{aligned}$$

We write $T \vdash F$ to indicate that the $\mathcal{L}_{Z_1}$-formula F is a logical consequence of T where T is one of the fragments defined above. We write $\vdash F$ to indicate that F follows from $\mathrm{BASIC}_{\mathrm{Z}_1}$ without any additional induction axiom.

3.2 The well-ordering proof in $\mathrm{I}\Sigma^0_n$

Let $\prec$ be a binary relation on ω definable by an $\mathcal{L}_{Z_1}$-formula. Let

$$\mathrm{field}(\prec) := \{n \in \omega : \exists m \in \omega (n \prec m \text{ or } m \prec n)\}.$$

For well-founded $\prec$ let

$$|n|_\prec := \sup\{|m|_\prec + 1 : m \prec n\} \in \Omega.$$

The *order-type of* $\prec$ is defined by

$$|| \prec || := \{|n|_\prec : n \in \omega\} \in \Omega.$$

Observe that $|n|_\prec = 0$ for all $n \notin \mathrm{field}(\prec)$.

We formalize the *notion of wellfoundedness*. Let

$$\begin{aligned} \mathit{Prog}(\prec, X) &:\equiv \forall x\, (\forall y\, (y \prec x \to y \in X) \to x \in X), \\ \mathit{Fund}(\prec, X) &:\equiv \mathit{Prog}(\prec, X) \to \forall x\, (x \in X). \end{aligned}$$

Then $\prec$ is well-founded if and only if $\mathbb{N} \models \mathit{Fund}(\prec, X)$ (observe that $\mathbb{N} \models \mathit{Prog}(\prec, X)$ always implies $\omega \setminus \mathrm{field}(\prec) \subset X$). Therefore, we say that T *proves the wellfoundedness of* $\prec$ iff $T \vdash \mathit{Fund}(\prec, X)$. Finally we define the *proof-theoretical ordinal* $\mathcal{O}(T)$ *of* T by

$$\mathcal{O}(T) :\equiv \sup\{|| \prec || : \prec \text{ is a primitive recursive definable binary relation and } T \vdash \mathit{Fund}(\prec, X)\}.$$

To compute $\mathcal{O}(\mathrm{I}\Sigma^0_n)$ we first give an upper estimation by adapting the well-ordering proof of Z_1 from [17]. We arithmetize the ordinals less than $\varepsilon_0 = \sup_{n<\omega} \omega_n(0)$ so that we can talk about them in $\mathcal{L}_{Z_1}$. Each ordinal $\alpha < \varepsilon_0$, $\alpha \neq 0$, can be written uniquely as $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ with $\alpha > \alpha_1 \geq \ldots \geq \alpha_n, n > 0$. This is called the CANTOR *normal form of* α and will be denoted by $\alpha =_{\mathrm{CNF}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. The following function $\ulcorner \cdot \urcorner : \varepsilon_0 \to \omega$ defined by transfinite recursion yields an arithmetization of ε_0. Let

$$\ulcorner \alpha \urcorner := \begin{cases} 0 & : \alpha = 0 \\ \langle \ulcorner \alpha_1 \urcorner, \ldots, \ulcorner \alpha_n \urcorner \rangle & : \alpha =_{\mathrm{CNF}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}, \end{cases}$$

where $\langle \ldots \rangle$ is a suitable primitive recursive coding function for finite sequences, e.g., the GÖDEL-numbers as defined in Chapter 6. Let $\mathcal{D}$ be the range of $\ulcorner \cdot \urcorner$, then $\ulcorner \cdot \urcorner : \varepsilon_0 \to \mathcal{D}$ is bijective. On $\mathcal{D}$ we define the relation $\prec$ and the functions $\hat{+}, \hat{\cdot}, \hat{\omega}^{\cdot}$ by

$$\begin{array}{rcl} \ulcorner \alpha \urcorner \prec \ulcorner \beta \urcorner & :\Longleftrightarrow & \alpha < \beta \\ \ulcorner \alpha \urcorner \hat{+} \ulcorner \beta \urcorner & := & \ulcorner \alpha + \beta \urcorner \\ \ulcorner \alpha \urcorner \hat{\cdot} \ulcorner \beta \urcorner & := & \ulcorner \alpha \cdot \beta \urcorner \\ \hat{\omega}^{\ulcorner \alpha \urcorner} & := & \ulcorner \omega^{\alpha} \urcorner \end{array}$$

$\mathcal{D}$, $\prec$, $\hat{+}$, $\hat{\cdot}$, $\hat{\omega}^{\cdot}$ are primitive recursive. For the rest of this chapter and the following two chapters let small Greek letters indicate ordinals resp. codes of ordinals.

On $\mathcal{D}$ the basic properties of ordinal arithmetic are provable in $\mathrm{I}\Sigma^0_0$. In particular we obtain:

3.2.1 Lemma $\mathrm{I}\Sigma^0_0$ *proves:*

$$\forall \alpha, \beta, \mu < \varepsilon_0 (\mu \neq 0 \land \alpha < \beta + \omega^{\mu} \to \exists \delta < \mu \exists n < \omega (\alpha < \beta + \omega^{\delta} \cdot n))$$

Proof: We argue informally in $\mathrm{I}\Sigma^0_0$. Fix $\alpha, \beta, \mu < \varepsilon_0$ with $\mu \neq 0$ and $\alpha < \beta + \omega^{\mu}$. If $\alpha \leq \beta$ the assertion is trivial, e.g., let $\delta = 0, n = 1$. So we may assume $\beta < \alpha$. We write α and β in their CANTOR normal forms – the CANTOR normal form of 0 is defined to be the empty sum.

$$\begin{array}{rcll} \alpha & =_{\mathrm{CNF}} & \omega^{\alpha_1} + \ldots + \omega^{\alpha_k} & (k > 0) \\ \beta & =_{\mathrm{CNF}} & \omega^{\beta_1} + \ldots + \omega^{\beta_l} & (l \geq 0) \end{array}$$

We distinguish the following cases according to the computation of $\beta < \alpha$:

1. If $l < k$ and $\alpha_1 = \beta_1, \dots, \alpha_l = \beta_l$, let $\delta := \alpha_{l+1}$ and $n := k+1$, hence

$$\beta + \omega^\delta \cdot n > \omega^{\alpha_1} + \ldots + \omega^{\alpha_l} + \underbrace{\omega^{\alpha_{l+1}} + \ldots + \omega^{\alpha_{l+1}}}_{k-l \text{ times}} \geq \alpha.$$

2. There is an $i \geq 0$ with $i < k$ and $\alpha_1 = \beta_1, \dots, \alpha_i = \beta_i$, $\alpha_{i+1} > \beta_{i+1}$. Assume $\mu \leq \alpha_{i+1}$, hence

$$\beta + \omega^\mu \leq \beta + \omega^{\alpha_{i+1}} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_{i+1}} \leq \alpha$$

contradicting $\alpha < \beta + \omega^\mu$. Hence $\delta := \alpha_{i+1} < \mu$. Let $n := k+1$, then

$$\beta + \omega^\delta \cdot n > \omega^{\alpha_1} + \ldots + \omega^{\alpha_i} + \underbrace{\omega^{\alpha_{i+1}} + \ldots + \omega^{\alpha_{i+1}}}_{k-i \text{ times}} \geq \alpha.$$

□

According to the previously defined canonical well-ordering of ε_0 we slightly modify the definition of *Prog* and *Fund*. Let

$$\begin{aligned}
\forall\alpha\, A(\alpha) &:\equiv \forall x\, (x \in \mathcal{D} \to A(x)),\\
\exists\alpha\, A(\alpha) &:\equiv \exists x\, (x \in \mathcal{D} \wedge A(x)),\\
\forall\beta{\leq}\alpha\, A(\beta) &:\equiv \forall\beta\, (\beta \preceq \alpha \to A(\beta)),\\
\exists\beta{\leq}\alpha\, A(\beta) &:\equiv \exists\beta\, (\beta \preceq \alpha \wedge A(\beta)),\\
\forall\beta{<}\alpha\, A(\beta) &:\equiv \forall\beta\, (\beta \prec \alpha \to A(\beta)),\\
\exists\beta{<}\alpha\, A(\beta) &:\equiv \exists\beta\, (\beta \prec \alpha \wedge A(\beta)),\\
\alpha \subset X &:\equiv \forall\beta{<}\alpha\, (\beta \in X)
\end{aligned}$$

where $\beta \preceq \alpha$ is an abbreviation for $\beta \prec \alpha \vee \beta = \alpha$. We define

$$\begin{aligned}
\mathit{Prog}(\alpha, X) &:\equiv \forall\beta{<}\alpha\, (\beta \subset X \to \beta \in X),\\
\mathit{Fund}(\alpha, X) &:\equiv \mathit{Prog}(\alpha, X) \to (\alpha \subset X).
\end{aligned}$$

If $A(x)$ and $F(X)$ are $\mathcal{L}_{Z_1}$-formulas, we write $F(A(\cdot))$ for substituting $t \in X$ by $A(t)$ and $t \notin X$ by $\neg A(t)$ in F.

In a first step we compute a lower bound of $\mathcal{O}(\mathrm{I}\Sigma^0_0)$.

3.2.2 Lemma *Let $A(x) \in \Pi^0_0$ and $l < \omega$. Then* $\mathrm{I}\Sigma^0_0 \vdash \mathit{Fund}(\omega{\cdot}l, A(\cdot))$.

Proof: We use induction on l. Let $X := A(\cdot)$.

If $l = 0$ then there is nothing to do because $\omega \cdot 0 = 0$ and $\vdash 0 \subset X$.

In the induction step $l \rightsquigarrow l+1$ the induction hypothesis yields

$$\mathrm{I}\Sigma^0_0 \vdash \mathit{Fund}(\omega \cdot l, X). \tag{3.1}$$

Now we argue in $\mathrm{I}\Sigma^0_0$. Assume

$$\mathit{Prog}(\omega \cdot (l+1), X) \equiv \forall \alpha{<}\omega \cdot (l+1)\, [\alpha \subset X \to \alpha \in X].$$

Hence $\mathit{Prog}(\omega \cdot l, X)$, which together with (3.1) yields

$$\omega \cdot l \subset X. \tag{3.2}$$

Now we show

$$\forall m{<}n\, (\omega \cdot l + m \in X),$$

which is a Σ^0_0-formula, for all $n < \omega$ by induction on n. Then we conclude $\forall n\, (\omega \cdot l + n \in X)$ and obtain $\omega \cdot (l+1) \subset X$ using (3.2). Hence $\mathit{Fund}(\omega \cdot (l+1), X)$.

For $n = 0$ there is nothing to do. In the induction step $n \rightsquigarrow n+1$ we use the induction hypothesis $\forall m{<}n\, (\omega \cdot l + m \in X)$ and (3.2) to obtain $\omega \cdot l + n \subset X$. Then $\mathit{Prog}(\omega \cdot (l+1), X)$ yields $\omega \cdot l + n \in X$, hence $\forall m{<}(n+1)\, (\omega \cdot l + m \in X)$. □

This lemma implies

$$\omega^2 = \sup\{\omega \cdot l \,:\, l < \omega\} \leq \mathcal{O}(\mathrm{I}\Sigma^0_0). \tag{3.3}$$

To compute a lower bound of $\mathcal{O}(\mathrm{I}\Sigma^0_{n+1})$ we have slightly more to do. We define the *jump of the set* X by

$$\mathit{Jp}(\alpha, X) := \{\beta \leq \alpha \,:\, \forall \gamma\, (\gamma + \omega^\beta \leq \omega^\alpha \wedge \gamma \subset X \to \gamma + \omega^\beta \subset X\}$$

and show the following lemma.

3.2.3 Lemma *Let $A(x) \in \Pi_{n+1}^0$. Then*

$$I\Sigma_{n+1}^0 \vdash Prog(\omega^\alpha, A(\cdot)) \to Prog(\alpha+1, Jp(\alpha, A(\cdot))).$$

Proof: Let $A(x) \in \Pi_{n+1}^0$. We argue in $I\Sigma_{n+1}^0$ and assume

$$Prog(\omega^\alpha, A(\cdot)), \tag{3.4}$$

$$\beta < \alpha + 1 \tag{3.5}$$

and

$$\beta \subset Jp(\alpha, A(\cdot)), \tag{3.6}$$

then we have to show that $\beta \in Jp(\alpha, A(\cdot))$. So assuming that γ satisfies

$$\gamma + \omega^\beta \le \omega^\alpha \tag{3.7}$$

and

$$\gamma \subset A(\cdot) \tag{3.8}$$

we have to conclude $\gamma + \omega^\beta \subset A(\cdot)$. If $\beta = 0$ we obtain $\gamma < \omega^\alpha$ using (3.7). Thus, (3.8) and (3.4) imply $A(\gamma)$, hence $\gamma + \omega^0 \subset A(\cdot)$. If $\beta > 0$ then Lemma 3.2.1 shows that for $\xi < \gamma + \omega^\beta$ there are $\delta < \beta$ and $k < \omega$ satisfying $\xi < \gamma + \omega^\delta \cdot k$. Thus, it suffices to prove

$$\gamma + \omega^\delta \cdot k \subset A(\cdot) \tag{3.9}$$

for any $\delta < \beta$ and $k < \omega$. To prove (3.9) we use induction on k. This is allowed in $I\Sigma_{n+1}^0$ because

$$\gamma + \omega^\delta \cdot k \subset A(\cdot) \;\equiv\; \forall \xi\, (\xi < \gamma + \omega^\delta \cdot k \to A(\xi))$$

is equivalent to a Π_{n+1}^0-formula and (Π_{n+1}^0-IND) is provable in $I\Sigma_{n+1}^0$. As $\gamma = \gamma + \omega^\delta \cdot 0$ we obtain $\gamma + \omega^\delta \cdot 0 \subset A(\cdot)$ by (3.8).

For the induction step $k \rightsquigarrow k+1$ we assume

$$\gamma + \omega^\delta \cdot k \subset A(\cdot). \tag{3.10}$$

As $\delta < \beta$ we obtain $\delta \in Jp(\alpha, A(\cdot))$ by (3.6). Together with the induction hypothesis (3.10) this yields

$$(\gamma + \omega^\delta \cdot k) + \omega^\delta \subset A(\cdot)$$

because $(\gamma+\omega^\delta\cdot k)+\omega^\delta = \gamma+\omega^\delta\cdot(k+1) < \gamma+\omega^{\delta+1} \leq \gamma+\omega^\beta \stackrel{(3.7)}{\leq} \omega^\alpha$, hence

$$\gamma+\omega^\delta\cdot(k+1) \subset A(\cdot).$$

□

3.2.4 Lemma *Let* $A(x) \in \Pi^0_{n+1}$. *Then*

$$\mathrm{I}\Sigma^0_{n+1} \vdash \mathit{Fund}(\alpha, \mathit{Jp}(\alpha, A(\cdot))) \to \mathit{Fund}(\omega^\alpha, A(\cdot)).$$

Proof: Assume

$$\mathit{Fund}(\alpha, \mathit{Jp}(\alpha, A(\cdot))) \tag{3.11}$$

and

$$\mathit{Prog}(\omega^\alpha, A(\cdot)), \tag{3.12}$$

then we have to show that $\omega^\alpha \subset A(\cdot)$. Lemma 3.2.3 applied to (3.12) gives us

$$\mathit{Prog}(\alpha+1, \mathit{Jp}(\alpha, A(\cdot))). \tag{3.13}$$

Then (3.11) together with (3.13) yields $\alpha \subset \mathit{Jp}(\alpha, A(\cdot))$. We obtain $\alpha \in \mathit{Jp}(\alpha, A(\cdot))$ using (3.13). Hence $\omega^\alpha \subset A(\cdot)$ and we are done. □

Let $\mathit{Jp}_0(\alpha, X) := X$, $\mathit{Jp}_{n+1}(\alpha, X) := \mathit{Jp}(\alpha, \mathit{Jp}_n(\omega^\alpha, X))$. We observe

$$A(x) \in \Pi^0_{n+1} \implies \mathit{Jp}(\alpha, A(\cdot)) \in \Pi^0_{n+2}.$$

Hence $X, \mathit{Jp}(\alpha_1, X), \ldots, \mathit{Jp}_n(\alpha_n, X) \in \Pi^0_{n+1}$, and Lemma 3.2.4 shows

$$\mathrm{I}\Sigma^0_{n+1} \vdash \mathit{Fund}(\alpha, \mathit{Jp}_{n+1}(\alpha, X)) \to \mathit{Fund}(\omega_{n+1}(\alpha), X).$$

Applying $\mathit{Prog}(l, A(\cdot))$ l-times to $0 \subset A(\cdot)$, we obtain $\vdash \mathit{Fund}(l, A(\cdot))$ for any $l < \omega$ and any $\mathcal{L}_{Z_1}$-formula $A(x)$.

Altogether this leads to $\mathrm{I}\Sigma^0_{n+1} \vdash \mathit{Fund}(\omega_{n+1}(l), X)$ for all $l < \omega$. Observe $\omega_{n+1}(\omega) = \omega_{n+1}(\omega^{\omega^0}) = \omega_{n+3}(0)$, hence

$$\omega_{n+3}(0) = \sup\{\omega_{n+1}(l) : l < \omega\} \leq \mathcal{O}(\mathrm{I}\Sigma^0_{n+1}). \tag{3.14}$$

Chapter 4

Semi-formal Systems

In this Chapter we introduce an infinitary language and an infinitary system which are convenient for investigating subsystems of pure number theory. We prove the familiar cut-elimination for the semi-formal system. Further we embed formal derivations of subsystems of pure number theory into this semi-formal system.

4.1 The infinitary language

First we repeat the definition and some basic facts of the *infinitary language* $\mathcal{L}_\infty$ from Pohlers [17].

The basic symbols of $\mathcal{L}_\infty$ are the logical symbols: countably many set variables $X_0, X_1, \ldots,\ \bigwedge, \bigvee, =, \neq, \in, \notin,$ and the same non-logical symbols as for $\mathcal{L}_{Z_1}$. The terms of $\mathcal{L}_\infty$ are the ground terms of $\mathcal{L}_{Z_1}$. *Prime formulas* or *atomic formulas* are the atomic Π^1_1-sentences of $\mathcal{L}_{Z_1}$. With these the $\mathcal{L}_\infty$-formulas are inductively defined using the following clause:

> If I is a non-empty index set and $(A_i)_{i\in I}$ is a sequence of $\mathcal{L}_\infty$-formulas, and all A_i contain no variable not in $X_0, \ldots, X_k$ for some $k < \omega$, then $\bigwedge(A_i)_{i\in I}$ and $\bigvee(A_i)_{i\in I}$ are $\mathcal{L}_\infty$-formulas.

In the sequel we will often write $\bigwedge_{i\in I} A_i$ and $\bigvee_{i\in I} A_i$ instead of $\bigwedge(A_i)_{i\in I}$ resp. $\bigvee(A_i)_{i\in I}$. Let F be an $\mathcal{L}_\infty$-formula. Assume that the variables occurring in F are among $Y_1, \ldots, Y_k$. Let $N_1, \ldots, N_k \in \mathfrak{P}(\omega)$. Then $\mathbb{N} \models F_{Y_1,\ldots,Y_k}[N_1, \ldots, N_k]$ is defined in the usual way[1]. Similar to $\mathcal{L}_{Z_1}$ we have no negation symbol in $\mathcal{L}_\infty$, but we can define a syntactic op-

[1]See [17] p. 24 for a definition.

eration $\neg : \mathcal{L}_\infty \to \mathcal{L}_\infty$ according to the DE MORGAN laws, see [17] p. 23.

From $\mathcal{L}_\infty$ we obtain sub-languages $\mathcal{L}_\Omega$ and $\mathcal{L}_\omega$ by restricting the index sets I in the inductive definition of the formulas to subsets of ω resp. finite subsets of ω. In the sequel we may assume that $I \in \omega \setminus \{0\}$ in the definition of $\mathcal{L}_\omega$.

The canonical *translation* $*$ of Π^1_1-sentences of $\mathcal{L}_{Z_1}$ to $\mathcal{L}_\Omega$ is given by the following inductive definition:

1. $F^* :\equiv F$ if F is an atomic formula
2. $(F_0 \wedge F_1)^* :\equiv \bigwedge_{i\leq 1} F_i^*$,
3. $(F_0 \vee F_1)^* :\equiv \bigvee_{i\leq 1} F_i^*$,
4. $(\forall x{\leq}t\, F(x))^* :\equiv \bigwedge_{n\leq t^{\mathbb{N}}} F(\underline{n})^*$ if $F \in \Delta^0_0$,
5. $(\exists x{\leq}t\, F(x))^* :\equiv \bigvee_{n\leq t^{\mathbb{N}}} F(\underline{n})^*$ if $F \in \Delta^0_0$,
6. $(\forall x\, F(x))^* :\equiv \bigwedge_{n<\omega} F(\underline{n})^*$ if $(\forall x\, F(x)) \notin \Delta^0_0$,
7. $(\exists x\, F(x))^* :\equiv \bigvee_{n<\omega} F(\underline{n})^*$ if $(\exists x\, F(x)) \notin \Delta^0_0$.

We define the *rank* $\mathrm{rk}(F)$ *of an $\mathcal{L}_\infty$-formula* F such that $F \in \mathcal{L}_\omega$ if and only if $\mathrm{rk}(F) < \omega$:

1. $\mathrm{rk}(F) := 0$ if F is atomic.
2. $\mathrm{rk}(\bigwedge_{i\in I} F_i) := \mathrm{rk}(\bigvee_{i\in I} F_i) := \max\{\mathrm{rk}(F_i)+1 : i \in I\}$
   if $\mathrm{card}(I) < \aleph_0$.
3. $\mathrm{rk}(\bigwedge_{i\in I} F_i) := \mathrm{rk}(\bigvee_{i\in I} F_i) := \sup(\{\mathrm{rk}(F_i)+1 : i \in I\} \cup \{\omega\})$
   if $\mathrm{card}(I) \geq \aleph_0$.

The definition is extended to $\mathcal{L}_{Z_1}$-formulas F by $\mathrm{rk}(F) := \mathrm{rk}(F^*)$. We observe

$$\begin{aligned}
F \in \mathcal{L}_\omega &\iff \mathrm{rk}(F) < \omega,\\
F \in \mathcal{L}_\Omega &\implies \mathrm{rk}(F) < \Omega,\\
F \in \mathcal{L}_{Z_1} &\implies \mathrm{rk}(F) < \omega + \omega,\\
F \in \Sigma^0_n &\implies \mathrm{rk}(F) < \omega + n.
\end{aligned}$$

4.2 The infinitary system

4.2.1 Definition *We inductively define the* infinitary system $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta$ *for ordinals* α, ρ *and* Δ *a finite set of* $\mathcal{L}_\infty$*-formulas by the following clauses.*

(Ax1) $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta, t = t$ *holds.*
$\mathop{|}\!\frac{\alpha}{\rho}\, \Delta, s \neq t$ *holds if* $s^{\mathbb{N}} \neq t^{\mathbb{N}}$.

(Ax2) $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta, s \in X, t \notin X$ *holds if* $s^{\mathbb{N}} = t^{\mathbb{N}}$.

($\bigwedge$) $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta, \bigwedge_{i \in I} F_i$ *holds if for all* $i \in I$ *there is some* $\alpha_i < \alpha$ *with* $\mathop{|}\!\frac{\alpha_i}{\rho}\, \Delta, F_i$.

($\bigvee$) $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta, \bigvee_{i \in I} F_i$ *holds if there is some* $\alpha_0 < \alpha$ *and* $i_0 \in I$ *with* $\mathop{|}\!\frac{\alpha_0}{\rho}\, \Delta, F_{i_0}$.

(Cut) $\mathop{|}\!\frac{\alpha}{\rho}\, \Delta$ *holds if there is some* $\alpha_0 < \alpha$ *and some* $\mathcal{L}_\infty$*-formula* F *with* $\mathrm{rk}(F) < \rho$ *and* $\mathop{|}\!\frac{\alpha_0}{\rho}\, \Delta, F$ *and* $\mathop{|}\!\frac{\alpha_0}{\rho}\, \Delta, \neg F$.

The infinitary system gives us a possibility to measure the complexity of true Π^1_1-sentences in the following sense: Using search-trees it is shown, e.g. in [17], that

$$\mathbb{N} \models F \iff \exists \alpha < \Omega \mathop{|}\!\frac{\alpha}{0}\, F^*$$

for Π^1_1-sentences F. Therefore, the *truth complexity of a* Π^1_1*-sentence* F is defined by

$$\mathrm{tc}(F) := \begin{cases} \min\{\alpha : \mathop{|}\!\frac{\alpha}{0}\, F^*\} & : \mathbb{N} \models F \\ \Omega & : \textit{otherwise.} \end{cases}$$

Before we can compute bounds for the truth complexities of Π^1_1-sentences which are provable in $\mathrm{I}\Sigma^0_n$ we have to fix a complete *formal system* for those theories. Let Φ be a set of $\mathcal{L}_{Z_1}$-formulas.

4.2.2 Definition *We inductively define the relation* $\mathrm{I}\Phi \vdash \Delta$ *for finite sets of* $\mathcal{L}_{Z_1}$*-formulas* Δ *by the following clauses.*

(Ax1) $\mathrm{I}\Phi \vdash \Delta$ *holds if* Δ *contains a mathematical axiom from the set* $\mathrm{BASIC}_{\mathrm{Z}_1}$.

(Ax2) $\mathrm{I}\Phi \vdash \Delta$ *holds if* Δ *contains an equality axiom of the form* $\forall x\, (x{=}x)$ *or* $\forall x\, \forall y\, (x{=}y \wedge A(x) \to A(y))$ *for atomic formulas* $A(x)$.

(Φ-IND) $\mathrm{I}\Phi \vdash \Delta$ *holds if* Δ *contains a formula of* (Φ-IND).

($\wedge$) $\mathrm{I}\Phi \vdash \Delta, F_0 \wedge F_1$ *holds if* $\mathrm{I}\Phi \vdash \Delta, F_i$ *for all* $i \in \{0,1\}$.

($\vee$) $\mathrm{I}\Phi \vdash \Delta, F_0 \vee F_1$ *holds if* $\mathrm{I}\Phi \vdash \Delta, F_i$ *for some* $i \in \{0,1\}$.

($\forall$) $\mathrm{I}\Phi \vdash \Delta, \forall x\, F(x)$ *holds if* $\mathrm{I}\Phi \vdash \Delta, F(y)$ *for some* y *which does not occur in* $\Delta, \forall x\, F(x)$.

($\exists$) $\mathrm{I}\Phi \vdash \Delta, \exists x\, F(x)$ *holds if* $\mathrm{I}\Phi \vdash \Delta, F(t)$ *for some* $\mathcal{L}_{Z_1}$*-term* t.

(Cut) $\mathrm{I}\Phi \vdash \Delta$ *holds if there is some* $\mathcal{L}_{Z_1}$*-formula* F *with* $\mathrm{I}\Phi \vdash \Delta, F$ *and* $\mathrm{I}\Phi \vdash \Delta, \neg F$.

We want to embed the derivable Π^1_1-sentences of $\mathrm{I}\Sigma^0_n$ into the infinitary system. To do that we need an *auxiliary infinitary system* $\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta$ for $\mathcal{L}_\infty$-formulas which, in addition to the clauses of $\mathrel{|\!\frac{\alpha}{\rho}} \Delta$, has the following kind of ω-rule:

(IND_n) $\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta, F(t)$ holds if $\mathrm{rk}(F(t)) < \omega + n$ and there is an $\alpha_0 < \alpha$ with $\mathrm{IND_n} \mathrel{|\!\frac{\alpha_0}{\rho}} \Delta, F(\underline{0})$ and $\mathrm{IND_n} \mathrel{|\!\frac{\alpha_0}{\rho}} \Delta, \neg F(\underline{k}), F(\underline{k+1})$ for all $k < t^{\mathbb{N}}$.

The basic properties of both infinitary systems are easily proved by induction on α:

Structural Rule $\mathrel{|\!\frac{\alpha}{\rho}} \Delta$ and $\alpha \le \alpha', \rho \le \rho', \Delta \subseteq \Delta' \implies \mathrel{|\!\frac{\alpha'}{\rho'}} \Delta'$.

$\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta$ and $\alpha \le \alpha', \rho \le \rho', \Delta \subseteq \Delta' \implies \mathrm{IND_n} \mathrel{|\!\frac{\alpha'}{\rho'}} \Delta'$.

($\bigwedge$)-Inversion $\mathrel{|\!\frac{\alpha}{\rho}} \Delta, \bigwedge_{i\in I} F_i \implies \mathrel{|\!\frac{\alpha}{\rho}} \Delta, F_i$ for all $i \in I$.

$\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta, \bigwedge_{i\in I} F_i$ and $\mathrm{rk}(\bigwedge_{i\in I} F_i) \ge \omega + n \implies \mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta, F_i$ for all $i \in I$.

($\bigvee$)-Exportation $\mathrel{|\!\frac{\alpha}{\rho}} \Delta, \bigvee_{i\le k} F_i \implies \mathrel{|\!\frac{\alpha}{\rho}} \Delta, F_0, \dots, F_k$.

$\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta, \bigvee_{i\le k} F_i$ and $\mathrm{rk}(\bigvee_{i\le k} F_i) \ge \omega + n$
$\implies \mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta, F_0, \dots, F_k$.

Equality Lemma $\mathrel{|\!\frac{\alpha}{\rho}} \Delta(s)$ and $s^{\mathbb{N}} = t^{\mathbb{N}} \implies \mathrel{|\!\frac{\alpha}{\rho}} \Delta(t)$.

$\mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta(s)$ and $s^{\mathbb{N}} = t^{\mathbb{N}} \implies \mathrm{IND_n} \mathrel{|\!\frac{\alpha}{\rho}} \Delta(t)$.

Using Structural Rules we can always assume – and in the sequel we will do so – that the conclusion of an inference is always included in the premise.

Some cuts in the infinitary systems can be eliminated. To do so we prove the following lemma:

4.2.3 Elimination Lemma *Let* $F \equiv \bigwedge_{i\in I} F_i$ *be an* $\mathcal{L}_\infty$*-formula and* $\mathrm{rk}(F) = \rho$.

$$\vdash^{\alpha}_{\rho} \Gamma, F \ \& \ \vdash^{\beta}_{\rho} \Delta, \neg F \ \& \ \rho > 0 \quad \Longrightarrow \quad \vdash^{\alpha+\beta}_{\rho} \Gamma, \Delta$$

$$\mathrm{IND_n} \vdash^{\alpha}_{\rho} \Gamma, F \ \& \ \mathrm{IND_n} \vdash^{\beta}_{\rho} \Delta, \neg F \ \& \ \rho \geq \omega + n \quad \Longrightarrow \quad \mathrm{IND_n} \vdash^{\alpha+\beta}_{\rho} \Gamma, \Delta$$

Proof: We use induction on β. (We only prove the first assertion, the second follows by a similar argument.) The interesting case is that $\neg F \equiv \bigvee_{i\in I} \neg F_i$ is the main formula of the last inference. Then the last inference has to be an application of $(\bigvee)$ (it cannot be (IND_n) in the second assertion as $\mathrm{rk}(F) \geq \omega + n$). Thus, there are some $\beta_0 < \beta$ and $i_0 \in I$ with $\vdash^{\beta_0}_{\rho} \Delta, \neg F, \neg F_{i_0}$. The induction hypothesis yields

$$\vdash^{\alpha+\beta_0}_{\rho} \Gamma, \Delta, \neg F_{i_0} \tag{4.1}$$

Applying $(\bigwedge)$-Inversion to $\vdash^{\alpha}_{\rho} \Gamma, F$ we obtain $\vdash^{\alpha}_{\rho} \Gamma, F_{i_0}$, hence

$$\vdash^{\alpha+\beta_0}_{\rho} \Gamma, \Delta, F_{i_0} \tag{4.2}$$

by a Structural Rule. An application of (Cut) to (4.1) and (4.2) yields $\vdash^{\alpha+\beta}_{\rho} \Gamma, \Delta$ as $\mathrm{rk}(F_{i_0}) < \mathrm{rk}(F) = \rho$ and $\alpha + \beta_0 < \alpha + \beta$. □

Using the Elimination Lemma we obtain the Elimination Theorem.

4.2.4 Elimination Theorem

$$\vdash^{\alpha}_{\rho+1} \Delta \ \& \ \rho > 0 \Longrightarrow \vdash^{2^\alpha}_{\rho} \Delta$$

$$\mathrm{IND_n} \vdash^{\alpha}_{\rho+1} \Delta \ \& \ \rho \geq \omega + n \Longrightarrow \mathrm{IND_n} \vdash^{2^\alpha}_{\rho} \Delta$$

Proof: The proof is by induction on α. □

4.3 The Embedding of $\mathrm{I}\Sigma^0_n$

4.3.1 Theorem *Let* $F(x_1, \ldots, x_k)$ *be an* $\mathcal{L}_{Z_1}$*-formula containing no variable not indicated. Assume* $\mathrm{I}\Sigma^0_n \vdash F$*, then there is an* $m < \omega$ *such that for all* $u_1, \ldots, u_k \in \omega$ $\vdash^{\omega\cdot m}_{\omega+n} [F(\underline{u_1}, \ldots, \underline{u_k})]^*$.

Proof: The proof is subdivided into three steps. First we embed the formal derivation into the auxiliary infinitary system $\mathrm{IND_n}$. Assuming $\mathrm{I}\Sigma_n^0 \vdash \Delta$ we show that there are some $m, r < \omega$ satisfying

$$\forall u_1, \dots, u_k \in \omega \quad \mathrm{IND_n} \mathrel{\vert\!\frac{m}{\omega+r}} [\Delta(\underline{u_1}, \dots, \underline{u_k})]^*$$

by induction on the definition of $\mathrm{I}\Sigma_n^0 \vdash \Delta$. The most interesting case is that Δ contains a (Σ_n^0-IND)-axiom, i.e., the universal closure of a formula

$$F(\underline{0}) \land \forall x\,(F(x) \to F(\mathrm{S}\,x)) \to \forall x\, F(x)$$

with $F(x) \in \Sigma_n^0$. By induction on the generation of F we can easily show that there is some $m < \omega$ such that

$$\mathrm{IND_n} \mathrel{\vert\!\frac{m}{0}} \neg F(\underline{l})^*, F(\underline{l})^*$$

for all $l < \omega$. Using the Equality Lemma, $(\bigwedge)$ and two times $(\bigvee)$ we derive from this

$$\mathrm{IND_n} \mathrel{\vert\!\frac{m+3}{0}} G, \neg F(\underline{l})^*, F(\underline{l+1})^*$$

where $G :\equiv [\neg F(\underline{0}) \lor \exists x\,(F(x) \land \neg F(\mathrm{S}\,x))]^*$. We also obtain

$$\mathrm{IND_n} \mathrel{\vert\!\frac{m+3}{0}} G, F(\underline{0})^*.$$

Hence

$$\mathrm{IND_n} \mathrel{\vert\!\frac{m+4}{0}} [\neg F(\underline{0}) \lor \exists x\,(F(x) \land \neg F(\mathrm{S}\,x))]^*, F(\underline{l})^*$$

for all $l < \omega$ using (IND_n). An inference $(\bigwedge)$ and two inferences $(\bigvee)$ yield the assertion.

By cut-elimination 4.2.4 we then obtain:

$$\forall u_1, \dots, u_k \in \omega \quad \mathrm{IND_n} \mathrel{\vert\!\frac{2_r(m)}{\omega+n}} [F(\underline{u_1}, \dots, \underline{u_k})]^*.$$

Embedding $\mathrm{IND_n}$ into the infinitary system yields

$$\forall u_1, \dots, u_k \in \omega \quad \mathrel{\vert\!\frac{\omega\cdot 2_r(m)}{\omega+n}} [F(\underline{u_1}, \dots, \underline{u_k})]^*$$

and we are done. For the last step we show the slightly more general assertion

$$\mathrm{IND_n} \mathrel{\vert\!\frac{\alpha}{\omega+n}} \Gamma \implies \mathrel{\vert\!\frac{\omega\cdot\alpha}{\omega+n}} \Gamma$$

by induction on α. The only interesting case is that the last inference was an application of (IND_n). The induction hypothesis applied to the premises of the inference leads to some $F(t)$ with $\mathrm{rk}(F(t)) < \omega + n$ and some $\alpha_0 < \alpha$ satisfying $\vdash^{\omega\cdot\alpha_0}_{\omega+n} \Gamma, F(\underline{0})$ and $\vdash^{\omega\cdot\alpha_0}_{\omega+n} \Gamma, \neg F(\underline{k}), F(\underline{k+1})$ for all $k < t^{\mathbb{N}}$. Then we can show $\vdash^{\omega\cdot\alpha_0+l}_{\omega+n} \Gamma, F(\underline{l})$ for $l \leq t^{\mathbb{N}}$ by inductively applying (Cut). Thus, as $\omega \cdot \alpha_0 + t^{\mathbb{N}} < \omega \cdot (\alpha_0 + 1) \leq \omega \cdot \alpha$, we obtain $\vdash^{\omega\cdot\alpha}_{\omega+n} \Gamma, F(t)$ using the Equality Lemma. $\square$

The Elimination Theorem applied to the last result yields

4.3.2 Corollary *Let F be a Π^1_1-sentence, $n > 0$ and $\mathrm{I}\Sigma^0_n \vdash F$, then there is an $m < \omega$ such that $\vdash^{\omega_n(m)}_{\omega} F^*$.*

Proof: The Embedding $\mathrm{I}\Sigma^0_n$-Theorem together with the Elimination Theorem leads to $\vdash^{2_n(\omega\cdot m)}_{\omega} F^*$ for some $m < \omega$. We compute $2^{\omega\cdot\alpha} = \omega^\alpha$ and $2^{(\omega^{1+\alpha})} = \omega^{(\omega^\alpha)}$. This yields $2_1(\omega \cdot \alpha) = \omega_1(\alpha)$ and $2_n(\omega \cdot (1+\alpha)) = \omega_n(\alpha)$ for $n > 1$ and $\alpha > 0$, thus the assertion follows. $\square$

Chapter 5

Upper Bounds for $\mathcal{O}(\mathrm{I}\Sigma_n^0)$

In the previous chapter we bounded the lengths of the infinitary derivations of $Fund(\prec, X)$. These derivations use cuts of translated Δ_0^0-formulas, which are $\mathcal{L}_\omega$-formulas. In this Chapter we connect the lengths of such derivations with the order-type of a well-founded arithmetical-definable binary and transitive relation $\prec$. We do this in two steps. First we prove the following cut-elimination for $\mathcal{L}_\omega$-formulas

$$\mathop{|\!\!-}\nolimits_{\omega}^{\alpha} \Delta \implies \mathop{|\!\!-}\nolimits_{1}^{\omega\cdot\alpha} \Delta.$$

Then we show the following Boundedness Theorem:

$$\mathop{|\!\!-}\nolimits_{1}^{\alpha} Fund(\prec, X) \implies || \prec || \leq \alpha.$$

Both results together yield

$$\mathop{|\!\!-}\nolimits_{\omega}^{\alpha} Fund(\prec, X) \implies || \prec || \leq \omega \cdot \alpha$$

and that is all we need to compute the missing estimations for $\mathcal{O}(\mathrm{I}\Sigma_n^0)$.

5.1 $\mathcal{L}_\omega$-cut-elimination

An $\mathcal{L}_\omega$-formula can be viewed as a finite tree whose leafs are labeled with atomic formulas and whose nodes are labeled with $\bigwedge$ and $\bigvee$. A heriditary inversion of such a formula is obtained by replacing each subtree above a node labeled with $\bigwedge$ by the subtree above one of its child-nodes. Any selection of this kind will be represented in form of a sequence.

5.1.1 Definition *For an $\mathcal{L}_\omega$-formula F we define a set of possible selection sequences $\mathrm{S}(F)$ and inversions F^f for $f \in \mathrm{S}(F)$. If F is atomic then let $\mathrm{S}(F) := \{\langle\rangle\}$ and $F^{\langle\rangle} :\equiv F$.*

In the case that $F \equiv \bigwedge_{i\le l} F_i$ we define

$$\mathrm{S}(F) := \{\langle j, g\rangle : j \le l \ \& \ g \in \mathrm{S}(F_j)\}$$

and we set $F^f :\equiv (F_j)^g$ for $f = \langle j, g\rangle \in \mathrm{S}(F)$.

In the remaining case that $F \equiv \bigvee_{i\le l} F_i$ we define

$$\mathrm{S}(F) := \{\langle g_0, \dots, g_l\rangle : g_0 \in \mathrm{S}(F_0), \dots, g_l \in \mathrm{S}(F_l)\}$$

and we set $F^f :\equiv \bigvee_{i\le l}(F_i)^{g_i}$ for $f = \langle g_0, \dots, g_l\rangle \in \mathrm{S}(F)$.

We give an example. Let $P_{i\,j}$, $j \le l_i$, $i \le l$, be atomic formulas. Let $F :\equiv \bigvee_{i\le l}\bigwedge_{j\le l_i} P_{i\,j}$. We compute

$$\begin{aligned}\mathrm{S}(F) &= \{\langle g_0, \dots, g_l\rangle : g_i \in \mathrm{S}\Big(\bigwedge_{j\le l_i} P_{i\,j}\Big) \text{ for } i \le l\}\\ &= \{\langle\langle j_0, \langle\rangle\rangle, \dots, \langle j_l, \langle\rangle\rangle\rangle : j_i \le l_i \text{ for } i \le l\}.\end{aligned}$$

Let $j_i \le l_i$ for $i \le l$, then

$$\begin{aligned}F^{\langle\langle j_0,\langle\rangle\rangle,\dots,\langle j_l,\langle\rangle\rangle\rangle} &\equiv \bigvee_{i\le l}\Big(\bigwedge_{j\le l_i} P_{i\,j}\Big)^{\langle j_i,\langle\rangle\rangle}\\ &\equiv \bigvee_{i\le l} P_{i\,j_i}\end{aligned}$$

Let F be an $\mathcal{L}_\omega$-formula. Let $Y_1, \dots, Y_k$ be the variables occurring in F and $N_1, \dots, N_k \in \mathfrak{P}(\omega)$. An easy induction on the generation of F shows

$$\Big(\forall f \in \mathrm{S}(F)\ \mathbb{N} \models F^f_{Y_1,\dots,Y_k}[N_1, \dots, N_k]\Big) \iff \mathbb{N} \models F_{Y_1,\dots,Y_k}[N_1, \dots, N_k].$$

Furthermore, we can show

5.1.2 Theorem (Heriditary Inversion) *If F is an $\mathcal{L}_\omega$-formula, then*

$$\left|\frac{\alpha}{\rho}\right.\Delta, F \implies \forall f \in \mathrm{S}(F)\ \left|\frac{\alpha}{\rho}\right.\Delta, F^f.$$

Proof: We use induction on α. If the main-formula of the last inference is not F, then an inference of the same kind (together with the induction hypothesis if $\alpha > 0$) yields the assertion. Otherwise, we distinguish the following cases:

Fix some $f \in \mathrm{S}(F)$. If F is atomic then $F^f \equiv F$. So there is nothing to do.

If $F \equiv \bigvee_{i\le l} F_i$ then the premise of the last inference is of the form $\vdash^{\alpha'}_{\rho} \Delta, F, F_j$ for some $j \le l$ and $\alpha' < \alpha$. Furthermore, $f = \langle g_0, \dots, g_l\rangle$ with $g_i \in \mathrm{S}(F_i)$ for $i \le l$. Applying the induction hypothesis twice we obtain $\vdash^{\alpha'}_{\rho} \Delta, F^f, F_j^{g_j}$, thus one $(\bigvee)$-inference yields $\vdash^{\alpha}_{\rho} \Delta, F^f$.

In the remaining case we have $F \equiv \bigwedge_{i\le l} F_i$, some $\alpha_i < \alpha$ and $\vdash^{\alpha_i}_{\rho} \Delta, F, F_i$ for $i \le l$. Then $f = \langle j, g\rangle$ with $j \le l$ and $g \in \mathrm{S}(F_j)$. We apply the induction hypothesis twice and obtain $\vdash^{\alpha_j}_{\rho} \Delta, F^f, F_j^g$. Thus, a Structural Rule yields $\vdash^{\alpha}_{\rho} \Delta, F^f$ observing $F^f \equiv F_j^g$. □

All these observations obviously extend to arbitrary $\mathcal{L}_\infty$-formulas. Of course the definition of the inversion then uses arbitrary selection trees which can be infinitary. The next result strongly depends on the finite structure of $\mathcal{L}_\omega$-formulas. It is the main observation in this section.

We define the *length,* $\mathrm{lh}(F)$*, of an* $\mathcal{L}_\omega$*-formula* F inductively by $\mathrm{lh}(A) := 1$ for an atomic formula A and

$$\mathrm{lh}(\bigwedge_{i\le l} F_i) := \mathrm{lh}(\bigvee_{i\le l} F_i) := \sum_{i\le l} \mathrm{lh}(F_i).$$

Obviously $0 < \mathrm{lh}(F) < \omega$ and $\mathrm{lh}(F) = \mathrm{lh}(\neg F)$. $\mathrm{lh}(F)$ counts the occurrences of atomic formulas in F.

5.1.3 $\mathcal{L}_\omega$-Cut-Elimination Lemma *Assume* $F \in \mathcal{L}_\omega$, $\rho > 0$, $\vdash^{\alpha}_{\rho} \Delta, F$ *and* $\vdash^{\alpha}_{\rho} \Delta, \neg F$. *Then* $\vdash^{\alpha+\mathrm{lh}(F)}_{\rho} \Delta$

Proof: With $\mathcal{L}_\omega$-inversion we obtain

$$\forall f \in \mathrm{S}(F) \qquad \vdash^{\alpha}_{\rho} \Delta, F^f \tag{5.1}$$

$$\forall g \in \mathrm{S}(\neg F) \qquad \vdash^{\alpha}_{\rho} \Delta, (\neg F)^g. \tag{5.2}$$

From this we prove $\vdash^{\alpha+\mathrm{lh}(F)}_{\rho} \Delta$ by induction on the generation of F.

If F is atomic, then $\mathrm{S}(\neg F) = \mathrm{S}(F) = \{\langle\rangle\}$, $(\neg F)^{\langle\rangle} \equiv \neg F$ and $F^{\langle\rangle} \equiv F$. As $\mathrm{rk}(F) = \mathrm{rk}(\neg F) = 0 < \rho$ and $\mathrm{lh}(F) = 1$ we obtain the assertion applying a (Cut).

In the case that F is not atomic we may assume $F \equiv \bigvee_{i\le l} F_i$, hence $\neg F \equiv \bigwedge_{i\le l} \neg F_i$. Now we prove for $j \le l+1$

$$\forall f_j \in \mathrm{S}(F_j) \dots \forall f_l \in \mathrm{S}(F_l) \quad \vdash^{\alpha+\sum_{0\le i<j} \mathrm{lh}(F_i)}_{\rho} \Delta, F_j^{f_j}, \dots, F_l^{f_l} \tag{5.3}$$

by induction on j. For $j = l+1$ this means an empty sequence of quantifiers $\forall f_i \in \mathrm{S}(F_i)$ and formulas $F_i^{f_i}$, hence $\vdash^{\alpha+\mathrm{lh}(F)}_{\rho} \Delta$.

For $j = 0$ we observe

$$\bigvee_{j\le i\le l} F_i^{f_i} \equiv \bigvee_{i\le l} F_i^{f_i} \equiv F^{\langle f_0,\dots,f_l\rangle}$$

$$\forall f_0 \in \mathrm{S}(F_0)\dots\forall f_l \in \mathrm{S}(F_l) \quad \Big(\langle f_0,\dots,f_l\rangle \in \mathrm{S}(F)\Big)$$

$$\sum_{0\le i<j} \mathrm{lh}(F_i) = 0,$$

therefore, (5.3) follows directly with $\bigvee$-Inversion from hypothesis (5.1).

In the induction step $j \rightsquigarrow j+1$, $j \le l$, we first fix $f_i \in \mathrm{S}(F_i)$ for $j < i \le l$. Let $L := \sum_{0\le i<j} \mathrm{lh}(F_i)$. The side induction hypothesis yields

$$\forall f \in \mathrm{S}(F_j) \quad \vdash^{\alpha+L}_{\rho} \Delta, F_{j+1}^{f_{j+1}},\dots,F_l^{f_l}, F_j^f \tag{5.4}$$

For any $g \in \mathrm{S}(\neg F_j)$ we know $\langle j,g\rangle \in \mathrm{S}(\neg F)$ and $(\neg F)^{\langle j,g\rangle} \equiv (\neg F_j)^g$, therefore, hypothesis (5.2) yields

$$\vdash^{\alpha}_{\rho} \Delta, (\neg F_j)^g$$

and by a Structural Rule we obtain

$$\forall g \in \mathrm{S}(\neg F_j) \quad \vdash^{\alpha+L}_{\rho} \Delta, F_{j+1}^{f_{j+1}},\dots,F_l^{f_l}, (\neg F_j)^g. \tag{5.5}$$

As F_j is a sub-formula of F we can apply the main induction hypothesis to (5.4) and (5.5) which yields $\vdash^{\alpha+L+\mathrm{lh}(F_j)}_{\rho} \Delta, F_{j+1}^{f_{j+1}},\dots,F_l^{f_l}$ which is the assertion (5.3) for $j+1$ as $L + \mathrm{lh}(F_j) = \sum_{0\le i<j+1} \mathrm{lh}(F_i)$. □

5.1.4 $\mathcal{L}_\omega$-Cut-Elimination Theorem

$$\vdash^{\alpha}_{\omega} \Delta \implies \vdash^{\omega\cdot\alpha}_{1} \Delta.$$

Proof by induction on α: The only interesting case, which is not immediate, is that $\vdash^{\alpha}_{\omega} \Delta$ is derived by a (Cut). Then there are some $\alpha_0 < \alpha$ and some $\mathcal{L}_\infty$-formula F with $\mathrm{rk}(F) < \omega$ and $\vdash^{\alpha_0}_{\omega} \Delta, F$ and $\vdash^{\alpha_0}_{\omega} \Delta, \neg F$. The induction hypothesis leads to $\vdash^{\omega\cdot\alpha_0}_{1} \Delta, F$ and $\vdash^{\omega\cdot\alpha_0}_{1} \Delta, \neg F$. From $\mathrm{rk}(F) < \omega$ we know $F \in \mathcal{L}_\omega$, hence $\vdash^{\omega\cdot\alpha_0+\mathrm{lh}(F)}_{1} \Delta$ applying the $\mathcal{L}_\omega$-Elimination Lemma. As $F \in \mathcal{L}_\omega$ we compute

$$\omega\cdot\alpha_0 + \mathrm{lh}(F) < \omega\cdot(\alpha_0+1) \le \omega\cdot\alpha.$$

□

5.2 The Boundedness Theorem

We can find a proof of $\left|\frac{\alpha}{0}\right. Fund(\prec, X) \implies ||\prec|| \leq 2^\alpha$ in [17] Theorem 13.10 – a result which goes back to GENTZEN. Nearly the same proof yields

$$\left|\frac{\alpha}{1}\right. Fund(\prec, X) \implies ||\prec|| \leq 2^\alpha.$$

Here we use a new idea to prove

$$\left|\frac{\alpha}{1}\right. Fund(\prec, X) \implies ||\prec|| \leq \alpha.$$

For this purpose we make some preliminary definitions and observations.

5.2.1 Definition *We define the negative points* $\mathrm{N}_X(\Delta)$ *of a set of* $\mathcal{L}_\infty$-*formulas* Δ *relative to a set-variable* X*:*

1. *If* F *is atomic let* $\mathrm{N}_X(F) := \begin{cases} \{s^{\mathbb{N}}\} & : F \equiv s \notin X \\ \emptyset & : otherwise \end{cases}$
2. $\mathrm{N}_X\left(\bigvee_{i\in I} F_i\right) := \mathrm{N}_X\left(\bigwedge_{i\in I} F_i\right) := \bigcup_{i\in I} \mathrm{N}_X(F_i)$
3. $\mathrm{N}_X(\Delta) := \bigcup_{F\in\Delta} \mathrm{N}_X(F)$

5.2.2 Lemma (Monotonicity) *Let* F *be an* $\mathcal{L}_\infty$-*formula containing no variable not in* $X, Y_1, \ldots, Y_k$*. Let* $M_1, M_2, N_1, \ldots, N_k \in \mathfrak{P}(\omega)$ *with* $\mathrm{N}_X(F) \subset M_1 \subset M_2$*. Then*

$$\mathbb{N} \models F_{X,Y_1,\ldots,Y_k}[M_1, N_1, \ldots, N_k] \implies \mathbb{N} \models F_{X,Y_1,\ldots,Y_k}[M_2, N_1, \ldots, N_k].$$

Proof: The proof is by induction on the generation of F. □

Let $\prec$ be a well-founded arithmetical definable binary and transitive relation. Its accessible part can be inductively defined by the *accessibility operator* $\mathrm{A}_\prec(S) := S \cup \{n \in \omega : \forall m \prec n(m \in S)\}$ for $S \subset \omega$. The α-th iteration of this operator is recursively defined by $\mathrm{A}^\alpha_\prec(S) := \mathrm{A}_\prec\left(S \cup \bigcup_{\beta<\alpha} \mathrm{A}^\beta_\prec(S)\right)$. Thus, we obtain the α-th stage of the inductive definition by $\mathrm{A}^\alpha_\prec(\emptyset)$.

In our further considerations we have to compute the effects of adjoining one element to S on $\mathrm{A}^\alpha_\prec(S)$. For this purpose we first give another, more direct description of $\mathrm{A}^\alpha_\prec(S)$.

The *enumeration function* of a class $\mathcal{O} \subset \mathrm{O\scriptstyle N}$ is defined by $\mathrm{en}_{\mathcal{O}}(\alpha) := \min\{\xi \in \mathcal{O} : (\forall \beta < \alpha)[\mathrm{en}_{\mathcal{O}}(\beta) < \xi]\}$. Let $\overline{\mathrm{en}}_{\mathcal{O}} := \mathrm{en}_{\mathrm{O\scriptstyle N} \setminus \mathcal{O}}$ be the *dual enumeration function* which enumerates the complement of $\mathcal{O}$. For $C \subset \omega$ let $C^{\prec} := \{|n|_{\prec} : n \in C\}$. Observe

$$C \subset C' \implies \overline{\mathrm{en}}_{C^{\prec}}(\alpha) \le \overline{\mathrm{en}}_{C'^{\prec}}(\alpha), \tag{5.6}$$

and

$$\overline{\mathrm{en}}_{(C \cup \{s\})^{\prec}}(\alpha) \le \overline{\mathrm{en}}_{C^{\prec}}(\alpha + 1). \tag{5.7}$$

We define

5.2.3 Definition *We define the* reachability operator *by*

$$\mathrm{R}^{\alpha}_{\prec}(C) := \{n \in \omega : |n|_{\prec} \le \overline{\mathrm{en}}_{C^{\prec}}(\alpha)\} \cup C.$$

Observe for $n \notin \mathrm{field}(\prec)$ that $n \in \mathrm{R}^{\alpha}_{\prec}(C)$ because $|n|_{\prec} = 0$. In the sequel we shortly write $\overline{\mathrm{en}}_{C,s}$ and $\mathrm{R}^{\alpha}_{\prec}(C, s)$ instead of $\overline{\mathrm{en}}_{C \cup \{s\}}$ and $\mathrm{R}^{\alpha}_{\prec}(C \cup \{s\})$.

For $n \in \omega$ with $|n|_{\prec} = \overline{\mathrm{en}}_{C^{\prec}}(\alpha)$ we have

$$(\forall x \prec n)(\exists \beta < \alpha)\,[x \in \mathrm{R}^{\beta}_{\prec}(C)]$$

and conversely if $(\forall x \prec n)(\exists \beta < \alpha)\,[x \in \mathrm{R}^{\beta}_{\prec}(C)]$ and $n \notin C$ then $(\exists \beta \le \alpha)\,[|n|_{\prec} = \overline{\mathrm{en}}_{C^{\prec}}(\beta)]$. Hence

$$\begin{aligned} \mathrm{R}^{\alpha}_{\prec}(C) &= C \cup \bigcup_{\beta < \alpha} \mathrm{R}^{\beta}_{\prec}(C) \cup \{n \in \omega : |n|_{\prec} = \overline{\mathrm{en}}_{C^{\prec}}(\alpha)\} \\ &= \mathrm{A}_{\prec}\Big(C \cup \bigcup_{\beta < \alpha} \mathrm{R}^{\beta}_{\prec}(C)\Big) \end{aligned}$$

By induction on α this yields

$$\mathrm{R}^{\alpha}_{\prec}(C) = \mathrm{A}^{\alpha}_{\prec}(C),$$

hence

$$(\forall x \prec n)\,[x \in \mathrm{R}^{\alpha}_{\prec}(C)] \implies n \in \mathrm{R}^{\alpha+1}_{\prec}(C). \tag{5.8}$$

The advantage of $\mathrm{R}^{\alpha}_{\prec}(C)$ in contrast to $\mathrm{A}^{\alpha}_{\prec}(C)$ is

$$\mathrm{R}^{\alpha}_{\prec}(C, s) \subset \mathrm{R}^{\alpha+1}_{\prec}(C) \cup \{s\} \tag{5.9}$$

which is obtained using (5.6) and (5.7).

In the sequel we consider the set variable X to be distinguished. Therefore, we can write $\mathrm{N}(F)$, $\mathbb{N} \models (\bigvee \Delta)[M]$ etc. instead of $\mathrm{N}_X(F)$, $\mathbb{N} \models (\bigvee \Delta)_X[M]$ etc.

5.2.4 Boundedness Lemma *Let X be the only variable occurring in Δ, then*

$$\mathop{\mid\!\!\!-}\nolimits_1^{\alpha} \neg Prog(\prec, X), \Delta \implies \mathbb{N} \vDash (\bigvee \Delta)\Big[\,\mathrm{R}_{\prec}^{\alpha}(\mathrm{N}(\Delta))\Big].$$

Proof: We use induction on α and consider several cases according to the last inference. In the case of an axiom already Δ itself is an axiom of the same kind and we are done. If the main formula of the last inference belongs to Δ then the assertion follows from the induction hypothesis, the Monotonicity Lemma and the correctness of the last inference.

We now turn to the interesting cases. If the main formula of the last inference is $\neg Prog(\prec, X)$, then we can find, using inversion, some $\alpha' < \alpha$ and some term s such that

$$\mathop{\mid\!\!\!-}\nolimits_1^{\alpha'} \neg Prog(\prec, X), \Delta, \forall x \prec s\,(x \in X) \tag{5.10}$$

and

$$\mathop{\mid\!\!\!-}\nolimits_1^{\alpha'} \neg Prog(\prec, X), \Delta, s \notin X. \tag{5.11}$$

If there is some $n \prec s$ such that $n \notin \mathrm{R}_{\prec}^{\alpha'}(\mathrm{N}(\Delta))$ then the induction hypothesis applied to (5.10) yields $\mathbb{N} \vDash (\bigvee \Delta)\Big[\,\mathrm{R}_{\prec}^{\alpha'}(\mathrm{N}(\Delta))\Big]$, and the assertion follows with the Monotonicity Lemma. Otherwise, (5.8) yields

$$s \in \mathrm{R}_{\prec}^{\alpha'+1}(\mathrm{N}(\Delta))$$

which together with (5.9) implies

$$\mathrm{R}_{\prec}^{\alpha'}(\mathrm{N}(\Delta), s) \subset \mathrm{R}_{\prec}^{\alpha'+1}(\mathrm{N}(\Delta)) \subset \mathrm{R}_{\prec}^{\alpha}(\mathrm{N}(\Delta)). \tag{5.12}$$

The induction hypothesis applied to (5.11) together with (5.12) entails

$$\mathbb{N} \vDash (\bigvee \Delta)\Big[\,\mathrm{R}_{\prec}^{\alpha}(\mathrm{N}(\Delta))\Big]$$

by the Monotonicity Lemma.

In the case that the last inference is a cut there are $\alpha' < \alpha$, an atomic formula F and premises

$$\mathop{\mid\!\!\!-}\nolimits_1^{\alpha'} \neg Prog(\prec, X), \Delta, F \tag{5.13}$$

and

$$\mathop{\mid\!\!\!-}\nolimits_1^{\alpha'} \neg Prog(\prec, X), \Delta, \neg F. \tag{5.14}$$

We may assume that F contains no other variable than X (otherwise F includes some variable Y different from X which can be substituted by X). Assume $F \equiv s \in X$. In the case $s \notin \mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))$ the induction hypothesis applied to (5.13) combined with the Monotonicity Lemma yields $\mathbb{N} \vDash \Big((\bigvee \Delta), s{\in}X\Big)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$, hence $\mathbb{N} \vDash (\bigvee \Delta)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$. Otherwise, $s \in \mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))$. The induction hypothesis applied to (5.14) leads to $\mathbb{N} \vDash (\bigvee \Delta)\Big[\mathrm{R}^{\alpha'}_{\prec}(\mathrm{N}(\Delta), s)\Big]$. With (5.9) we observe

$$\mathrm{R}^{\alpha'}_{\prec}(\mathrm{N}(\Delta), s) \subset \mathrm{R}^{\alpha'+1}_{\prec}(\mathrm{N}(\Delta)) \cup \{s\} \subset \mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta)),$$

so using the Monotonicity Lemma we obtain $\mathbb{N} \vDash (\bigvee \Delta)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$. If $F \equiv s \notin X$ the situation is quite symmetrical. In the remaining case F is an atomic sentence not of the form $s \in X$ or $s \notin X$. Then the induction hypothesis applied to (5.13) and (5.14) combined with the Monotonicity Lemma yields $\mathbb{N} \vDash \Big((\bigvee \Delta), F\Big)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$ and $\mathbb{N} \vDash \Big((\bigvee \Delta), \neg F\Big)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$, hence $\mathbb{N} \vDash (\bigvee \Delta)\Big[\mathrm{R}^{\alpha}_{\prec}(\mathrm{N}(\Delta))\Big]$. □

5.2.5 Boundedness Theorem

$$\vdash^{\alpha}_{1} \mathit{Fund}(\prec, X) \implies ||\prec|| \le \alpha.$$

Proof: First we observe that there is an $\alpha' < \alpha$ such that

$$\vdash^{\alpha'}_{1} \neg \mathit{Prog}(\prec, X), \forall x\,(x \in X). \tag{5.15}$$

To obtain this we show by induction on β:

If $\beta > 0$ and $P_1, \ldots, P_k$ are atomic formulas satisfying

$$\vdash^{\beta}_{1} \mathit{Fund}(\prec, X), P_1, \ldots, P_k,$$

then there is an $\gamma < \beta$ such that

$$\vdash^{\gamma}_{1} \mathit{Prog}(\prec, X), \forall x\,(x \in X), P_1, \ldots, P_k.$$

If $\vdash^{\beta}_{1} \mathit{Fund}(\prec, X), P_1, \ldots, P_k$ by an axiom let $\gamma = 0$. If the main formula of the last inference is $\mathit{Fund}(\prec, X)$ then we are in the situation of an $(\bigvee)$-inference, and we obtain the assertion by $(\bigvee)$-Exportation. If the last inference was a cut then there is a prime formula P and some $\beta_0 < \beta$ with

$$\vdash^{\beta_0}_{1} \mathit{Fund}(\prec, X), P_1, \ldots, P_k, P$$

and

$$\vdash^{\beta_0}_{1} Fund(\prec, X), P_1, \dots, P_k, \neg P.$$

If $\beta_0 > 0$ then the induction hypothesis and a (Cut) yield

$$\vdash^{\beta_0}_{1} Prog(\prec, X), \forall x\, (x \in X), P_1, \dots, P_k.$$

Otherwise, $P_1, \dots, P_k$ has to be an axiom and we obtain

$$\vdash^{0}_{1} Prog(\prec, X), \forall x\, (x \in X), P_1, \dots, P_k.$$

We compute $\mathrm{N}_X \Big(\forall x\, (x \in X)\Big) = \emptyset$ and $\overline{\mathrm{en}}_\emptyset(\alpha') = \alpha'$. So the previously proved Boundedness Lemma applied to (5.15) yields

$$\forall x \quad x \in \mathrm{R}^{\alpha'}_{\prec}(\emptyset),$$

hence $\forall x\, (|x|_\prec \leq \alpha')$, hence $|| \prec || = \{|n|_\prec \,:\, n \in \omega\} \subset \alpha' + 1 \leq \alpha$. □

5.3 Applications: $\mathcal{O}(\mathrm{I}\Sigma^0_{n+1}) = \omega_{n+3}(0)$ and $\mathcal{O}(\mathrm{I}\Sigma^0_0) = \omega^2$

In the last part of this chapter we use the $\mathcal{L}_\omega$-Cut-Elimination Theorem and the Boundedness Theorem to compute $\mathcal{O}(\mathrm{I}\Sigma^0_n)$.

Assume $\mathrm{I}\Sigma^0_{n+1} \vdash Fund(\prec, X)$. Using Corollary 4.3.2 there is an $m < \omega$ such that $\vdash^{\omega_{n+1}(m)}_{\omega} Fund(\prec, X)^*$. Now the $\mathcal{L}_\omega$-Elimination Theorem 5.1.4 yields $\vdash^{\omega\cdot\omega_{n+1}(m)}_{1} Fund(\prec, X)^*$. Then the Boundedness Theorem 5.2.5 yields $|| \prec || \leq \omega \cdot \omega_{n+1}(m) < \omega_{n+1}(\omega) = \omega_{n+3}(0)$. Thus, we have shown $\mathcal{O}(\mathrm{I}\Sigma^0_{n+1}) \leq \omega_{n+3}(0)$. Together with the result (3.14) from Chapter 3 this yields

5.3.1 Corollary $\mathcal{O}(\mathrm{I}\Sigma^0_{n+1}) = \omega_{n+3}(0)$. □

Assume $\mathrm{I}\Sigma^0_0 \vdash Fund(\prec, X)$. With the Compactness Theorem and the Deduction Theorem for first order logic there are $A_1, \dots, A_k \in (\Sigma^0_0\text{-IND})$ such that

$$\vdash \neg A_1, \dots, \neg A_k, Fund(\prec, X)$$

As $A(\underline{0}) \wedge \forall x\,(A(x) \to A(\mathrm{S}\,x)) \to \forall x\,A(x)$ is logically equivalent to $\forall x\,I_A(x)$, where $I_A(x) :\equiv A(\underline{0}) \wedge \forall y{<}x\,(A(y) \to A(\mathrm{S}\,y)) \to A(x)$, there are Δ^0_0-formulas $F_1, \ldots, F_k$ such that

$$\vdash \neg\forall x\,I_{F_1}(x), \ldots, \neg\forall x\,I_{F_k}(x), Fund(\prec, X). \tag{5.16}$$

Actually we have to consider universal closures of formulas I_F. But by coding techniques we may always assume that the length of the block of universal quantifiers is 1.

We need a slightly modified definition of the rank function ($\mathrm{rk}(F) := 0$ if F is atomic and $\mathrm{rk}(\bigwedge_{i\in I} F_i) := \mathrm{rk}(\bigvee_{i\in I} F_i) := \sup\{\mathrm{rk}(F_i)+1 : i \in I\}$) in order to produce from the above formal derivation a finitary cut-free semi-formal derivation. We can directly embed derivation 5.16 into a modified infinitary system (which uses the new rank definition) obtaining some $m < \omega$ and $r < \omega$ such that

$$\vdash^{m}_{r} \neg(\forall x\,I_{F_1}(x))^*, \ldots, \neg(\forall x\,I_{F_k}(x))^*, Fund(\prec, X)^*.$$

Adapting the Elimination Theorem 4.2.4 leads to

$$\vdash^{2_r(m)}_{0} \neg(\forall x\,I_{F_1}(x))^*, \ldots, \neg(\forall x\,I_{F_k}(x))^*, Fund(\prec, X)^*.$$

To obtain

$$\vdash^{\omega\cdot 2_r(m)}_{1} Fund(\prec, X)^* \tag{5.17}$$

we prove

$$\vdash^{\alpha}_{0} \neg(\forall x\,I_{F_1}(x))^*, \ldots, \neg(\forall x\,I_{F_k}(x))^*, \Delta \implies \vdash^{\omega\cdot\alpha}_{1} \Delta$$

by induction on α. The assertion follows directly (with the induction hypothesis if $\alpha > 0$) if the main formula of the last inference was not $\neg(\forall x\,I_{F_i}(x))^*$ for $i \in \{1, \ldots, k\}$. Otherwise, we can find some $\alpha_0 < \alpha$ and some $l \in \omega$ such that

$$\vdash^{\alpha_0}_{0} \neg(\forall x\,I_{F_1}(x))^*, \ldots, \neg(\forall x\,I_{F_k}(x))^*, \Delta, \neg I_{F_i}(\underline{l})^*.$$

Using the induction hypothesis we obtain

$$\vdash^{\omega\cdot\alpha_0}_{1} \Delta, \neg I_{F_i}(\underline{l})^*.$$

Adapting the embedding of induction from the proof of Theorem 4.3.1 we observe that there are some $m', r' < \omega$ with $\vdash^{m'}_{r'} I_{F_i}(\underline{l})^*$, hence

$$\vdash^{2_{r'}(m')}_{0} \Delta, I_{F_i}(\underline{l})^*.$$

Obviously $I_{F_i}(\underline{l}) \in \Delta^0_0$, hence $I_{F_i}(\underline{l})^* \in \mathcal{L}_\omega$ and the $\mathcal{L}_\omega$-Elimination Lemma yields

$$\Big|\frac{\omega\cdot\alpha_0+2_{r'}(m')+\mathrm{lh}[I_{F_i}(\underline{l})^*]}{1}\ \Delta.$$

We compute $\omega \cdot \alpha_0 + 2_{r'}(m') + \mathrm{lh}[I_{F_i}(\underline{l})^*] < \omega \cdot (\alpha_0 + 1) \leq \omega \cdot \alpha$.

The Boundedness Theorem applied to (5.17) yields

$$|| \prec || < \omega \cdot 2_r(m) < \omega \cdot \omega = \omega^2,$$

hence $\mathcal{O}(\mathrm{I}\Sigma^0_0) \leq \omega^2$. Together with the result (3.3) from the middle of chapter 3 this yields

5.3.2 Corollary $\mathcal{O}(\mathrm{I}\Sigma^0_0) = \omega^2$. □

Chapter 6

Notations for Exponentiation

A necessary condition for a function f to be feasibly computable is that it grows at most polynomially, i.e., it has *polynomial growth rate*[1], which means that there is a polynomial q_f such that $(\forall n)[|f(n)| \leq q_f(|n|)]$ – a condition which is satisfied, e.g., by all functions from the polynomial hierarchy **PH**, in particular by the polytime functions. Therefore, it is difficult to deal with the exponentiation function directly in the investigation on bounded arithmetic theories. One possibility of dealing with exponentiation is shown for example in [12] that the graph of the exponentiation function can be defined by a Δ_0^0-formula.

In this thesis we will follow another idea. In the ordinal analysis of Z_1 we coded ordinals less than ε_0 in such a way that basic operations like $+$, $\cdot$ and $\lambda\alpha.\omega^\alpha$ on the ordinal notations became primitive recursive functions (cf. Chapter 3). Replacing ω by 2 yields a coding of the natural numbers in such a way that some basic arithmetical operations like $+$, $\lambda n.2 \cdot n$ and *exponentiation* $\lambda n.2^n$ on this notations become polytime operations.

6.1 Exponential codes for natural numbers

Let $\langle \ldots \rangle$ be the GÖDEL *numbers* for sequences as defined in [6] p. 8 with the change that we do not reverse the order of the bits. The following equations define such a coding. First we define a function $s * a$ for $s, a \in \omega$ by limited recursion on the notation of a. This function adds

[1] cf. [6] p. 9.

the value a to the sequence s.

$$\begin{aligned} s * 0 = (s0010)_2 &= 16 \cdot s + 2 \\ s * 1 = (s0011)_2 &= 16 \cdot s + 3 \\ s * (ai)_2 = ((s * a)1i)_2 &= 4 \cdot (s * a) + 2 + i, \quad (i = 0, 1 \text{ and } a \neq 0). \end{aligned}$$

Then the GÖDEL numbers are given by

$$\begin{aligned} \langle\rangle &= 0 \\ \langle a_1, \dots, a_k, a_{k+1}\rangle &= \langle a_1, \dots, a_k\rangle * a_{k+1}. \end{aligned}$$

Let `Seq` be the polytime *set of all* GÖDEL *numbers.*

How does GÖDEL numbering work? The GÖDEL number for the sequence $a_1, \dots, a_k$ is constructed as follows. First write the a_i's in binary notation so we obtain a string of 0's, 1's and commas. Then we replace each 0 by "10", each 1 by "11" and each comma by "00". The resulting string of zeros and ones is the binary representation of the GÖDEL number $\langle a_1, \dots, a_k\rangle$. For example the GÖDEL number of $3, 4, 5$ is $(11110011101000111011)_2$ or 997.947. $\langle\rangle$ is defined to be 0.

In the following we introduce some polytime functions which manipulate GÖDEL numbers.

$$\begin{aligned} \langle a_1, \dots, a_k\rangle ** \langle b_1, \dots, b_l\rangle &= \langle a_1, \dots, a_k, b_1, \dots, b_l\rangle \\ \beta(0, \langle a_1, \dots, a_k\rangle) &= k \\ \mathrm{lh}(\langle a_1, \dots, a_k\rangle) &= k \\ \beta(i+1, \langle a_1, \dots, a_k\rangle) &= a_{i+1}, \quad i < k \\ \mathrm{trunc_r}(\langle a_1, \dots, a_k, a_{k+1}\rangle) &= \langle a_1, \dots, a_k\rangle \\ \mathrm{trunc_l}(\langle a_1, a_2, \dots, a_k\rangle) &= \langle a_2, \dots, a_k\rangle \\ \mathrm{first}(\langle a_1, \dots, a_k\rangle) &= a_1 \\ \mathrm{last}(\langle a_1, \dots, a_k\rangle) &= a_k \\ \mathrm{SqBd}(k, l) &= (k \# \mathrm{S}_1(\mathrm{S}_1(l)))^2. \end{aligned}$$

$\mathrm{SqBd}(\cdot, \cdot)$ has the property

$$\forall a_1, \dots, a_{|k|} \leq l \ \Big(\langle a_1, \dots, a_{|k|}\rangle \leq \mathrm{SqBd}(k, l)\Big).$$

In the sequel we use small Greek letters for natural numbers that are interpreted as exponential notations. Using this coding function we define

$$\begin{aligned} \hat{0} &:= \langle\rangle \\ \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k} &:= \langle\alpha_1,\ldots,\alpha_k\rangle \\ \hat{1} &:= \check{2}^{\hat{0}}. \end{aligned}$$

The intended meaning of these codes becomes clear from the evaluation function which is given by

$$\begin{aligned} \Phi(\hat{0}) &= 0 \\ \Phi(\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}) &= 2^{\Phi(\alpha_1)}+\ldots+2^{\Phi(\alpha_k)}, \end{aligned}$$

thus $\Phi(\hat{1}) = 1$. Of course Φ is not a polytime function.

Now we define the predicates $\mathcal{E}$, $\prec$ and the functions $\Phi_{\mathcal{E}}$, $\mathrm{T}_{\mathcal{E}}$ by the following equations:

$$\begin{aligned} \alpha \in \mathcal{E} &\iff \alpha = 0 \text{ or there are } \alpha_1,\ldots,\alpha_k \in \mathcal{E} \text{ with} \\ & \alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k} \text{ and } \Phi(\alpha_k) < \ldots < \Phi(\alpha_1) \\ \Phi_{\mathcal{E}} &:= \Phi \restriction \mathcal{E} \\ \alpha \prec \beta &\iff \alpha,\beta \in \mathcal{E} \;\&\; \Phi_{\mathcal{E}}(\alpha) < \Phi_{\mathcal{E}}(\beta) \\ \mathrm{T}_{\mathcal{E}} &:= {\Phi_{\mathcal{E}}}^{-1}. \end{aligned}$$

For $\alpha, \beta \in \mathcal{E}$ we give an implicit definition of the functions $\hat{+}$ and $\hat{2}$:

$$\begin{aligned} \Phi_{\mathcal{E}}(\alpha \,\hat{+}\, \beta) &= \Phi_{\mathcal{E}}(\alpha) + \Phi_{\mathcal{E}}(\beta) \\ \Phi_{\mathcal{E}}(\hat{2}^{\alpha}) &= 2^{\Phi_{\mathcal{E}}(\alpha)}. \end{aligned}$$

$\mathcal{E}$ is the *set of exponential notations*. In the rest of this Chapter we show that the predicates $\mathcal{E}$, $\prec$ and the functions $\hat{+}$, $\hat{2}$, $\mathrm{T}_{\mathcal{E}}$ are polytime. First we observe that the desired exponentiation function on $\mathcal{E}$ can be written simply as $\lambda\alpha.\,\hat{2}^{\alpha} := \langle\alpha\rangle$. Therefore, $\hat{2}$ is a polytime function. Let

$$f_i(n) := \underbrace{\hat{2}(\ldots\hat{2}}_{i\text{-times}}(\mathrm{T}_{\mathcal{E}}(n))\ldots)$$

then we compute

$$\Phi_{\mathcal{E}}(f_i(n)) = 2^{\cdot^{\cdot^{2^n}}} \Big\} \; i\text{-times}.$$

After having seen that $\mathrm{T}_{\mathcal{E}}$ is polytime this shows that $\Phi_{\mathcal{E}}$ cannot be polytime.

6.2 Limited course-of-values recursion

The verification that the predicates $\mathcal{E}, \prec$ and the functions $\hat{+}, \mathrm{T}_{\mathcal{E}}$ are polytime requires a special limited course-of-values recursion.

In the sequel we will use limited recursion (on notation) to define polytime functions. In doing so we often use $\mathrm{lh}(s)$ to bound recursion. This is allowed since $\mathrm{lh}(s) \leq |s|$.

The usual course-of-values recursion is equivalent to primitive recursion, thus, in general, polytime functions are not closed under this rule. Another, more technical, aspect is that $\lambda n.\langle 0, 1, \ldots, n-1\rangle$ growths exponentially, because for $n > 0$ we compute $|\langle 0, 1, \ldots, n-1\rangle| \geq 2 \cdot n > n$, hence $\langle 0, 1, \ldots, n-1\rangle \geq 2^n$. Therefore, one requirement of limited course-of-values recursion is that the course is given by a polytime function.

In the following let $s \sqsubset t$ mean that s, t are GÖDEL numbers and s is a subsequence of t, i.e., if $\mathrm{lh}(s) = k$ and $t = \langle t_0, \ldots, t_{l-1}\rangle$ then $k \leq l$ and

$$\exists i_0, \ldots, i_{k-1} \ \Big(i_0 < \ldots < i_{k-1} < l \ \& \ s = \langle t_{i_0}, \ldots, t_{i_{k-1}}\rangle\Big).$$

6.2.1 Definition *A* course-function *is a function* course$(\cdot)$ *satisfying*

$$\mathrm{course}(s) \sqsubset \langle 0, \ldots, s-1\rangle$$

and

$$\mathrm{course}(s) = \langle s_0, \ldots, s_{k-1}\rangle \implies \forall i{<}k \ \Big(\mathrm{course}(s_i) \sqsubset \langle s_0, \ldots, s_{i-1}\rangle\Big).$$

The course-of-values of a function f according to course$(\cdot)$ *is defined by*

$$f^{\mathrm{course}}(s) := \langle f(s_0), \ldots, f(s_{k-1})\rangle$$

provided that $\mathrm{course}(s) = \langle s_0, \ldots, s_{k-1}\rangle$.

If f and course$(\cdot)$ are polytime then also f^{course} is polytime. This can be seen, using limited recursion, by a similar argument as in the following theorem.

6.2.2 Theorem (limited course-of-values recursion)
Let course$(\cdot)$ *be a course-function. Given a function g there exists a uniquely defined function f solving*

$$f(s) = g(s, f^{\mathrm{course}}(s)).$$

If in addition course($\cdot$) *and* g *are polytime and there exists another polytime function* h *satisfying*

$$f(s) \leq h(s)$$

then this f *is polytime, too.*

Proof: Existence and uniqueness are proved as usual. For the second part of the theorem we define the function

$$\mathrm{select}(\langle a_0, \ldots, a_{k-1}\rangle, \langle a_{i_1}, \ldots, a_{i_r}\rangle, \langle b_0, \ldots, b_{l-1}\rangle) := \langle b_{i_1}, \ldots, b_{i_r}\rangle$$

for an increasing sequence $\langle a_0, \ldots, a_{k-1}\rangle$, $i_1 < \ldots < i_r < \min(k, l)$. Using functions

$$\mathrm{b}(x) := \begin{cases} \langle \alpha, \beta, \gamma, \delta * c\rangle & : x = \langle \alpha * a, \beta * a, \gamma * c, \delta\rangle \\ \langle \alpha, \beta * b, \gamma, \delta\rangle & : x = \langle \alpha * a, \beta * b, \gamma * c, \delta\rangle \text{ and } a \neq b \\ x & : \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \mathrm{r}(\langle a_1, \ldots, a_k\rangle) &:= \langle a_k, \ldots, a_1\rangle \\ \mathrm{select}(\alpha, \beta, s) &:= \beta(4, \mathrm{b}^{\mathrm{lh}(\alpha)}(\langle \mathrm{r}(\alpha), \mathrm{r}(\beta), \mathrm{r}(s), \langle\rangle\rangle)) \leq s \end{aligned}$$

we observe that select($\cdot$) is polytime by limited recursion. Here $\mathrm{b}^x(a)$ is the x-fold iteration of $b(\cdot)$ applied to a.

In order to prove the assertion it suffices to show that f^{course} is polytime. Let $t = \mathrm{course}(s) = \langle b_0, \ldots, b_{l-1}\rangle$, then we define a polytime function $\phi(t) = \langle f(b_0), \ldots, f(b_{l-1})\rangle$ with the use of $\tilde{\phi}(t, i) = \langle f(b_0), \ldots, f(b_{i-1})\rangle$ and the fact that course(b_i) is a subsequence of $\langle b_0, \ldots, b_{i-1}\rangle$. Then we can compute for $i < l$

$$\begin{aligned} f(b_i) &= g(b_i, f^{\mathrm{course}}(b_i)) \\ &= g(b_i, \mathrm{select}(t, \mathrm{course}(b_i), \tilde{\phi}(t, i))). \end{aligned}$$

We define

$$\begin{aligned} \tilde{\phi}(t, 0) &:= \langle\rangle \\ \tilde{\phi}(t, i+1) &:= \tilde{\phi}(t, i) * g(\beta(i+1, t), \\ &\qquad\qquad \mathrm{select}(t, \mathrm{course}(\beta(i+1, t)), \tilde{\phi}(t, i))) \\ \phi(t) &:= \tilde{\phi}(t, \mathrm{lh}(t)) \leq h^{\mathrm{course}}(t) \\ f^{\mathrm{course}}(s) &:= \phi(\mathrm{course}(s)). \end{aligned}$$

By limited recursion f^{course} is polytime. □

6.3 $\mathcal{E}$, $\prec$, $\hat{+}$, $\mathrm{T}_{\mathcal{E}}$ are polytime

We need some special course functions which compute those subsequences such that all values are included which are needed in the definition of $\mathcal{E}$, $\prec$, $\hat{+}$ and $\mathrm{T}_{\mathcal{E}}$. We start defining

$$\mathrm{sort}(\langle a_1, \dots, a_k \rangle) \;:=\; \langle b_1, \dots, b_l \rangle$$

where $\{a_1, \dots, a_k\} = \{b_1, \dots, b_l\}$ and $b_1 < \dots < b_l$. $\mathrm{sort}(\cdot)$ can be computed using one of the commonly known sorting algorithms, e.g., one which runs in time $O(n^2)$ sorting n objects. Thus, $\mathrm{sort}(s)$ is computable in time $O(|s|^2)$, hence polytime.

Now we define

$$\mathrm{U}(\langle\langle a_{1\,1}, \dots, a_{1\,i_1}\rangle, \dots, \langle a_{k\,1}, \dots, a_{k\,i_k}\rangle\rangle) \;:=\; \langle b_1, \dots, b_l \rangle$$

where $b_1 < \dots < b_l$ and

$$\{b_1, \dots, b_l\} \;=\; \{a_{1\,1}, \dots, a_{1\,i_1}, \dots, a_{k\,1}, \dots, a_{k\,i_k}\}.$$

The following equations may be used to observe that $\mathrm{U}(\cdot)$ is polytime. Let $s = \langle s_0, \dots, s_{k-1} \rangle$.

$$\begin{aligned} f(\langle s_0, \dots, s_{k-1} \rangle) \;&:=\; s_0 \mathbin{**} \dots \mathbin{**} s_{k-1} \;\leq\; \mathrm{SqBd}(s, s) \\ \mathrm{U}(s) \;&:=\; \mathrm{sort}(f(s)). \end{aligned}$$

By limited recursion f is polytime, thus also $\mathrm{U}(\cdot)$. We use these functions to see that the *transitive closure*[2] of a sequence can be computed by a polytime function. To this end, observe that $\mathrm{U}^{|s|}(s) = \langle\rangle$ and let

$$g(s) \;:=\; s \mathbin{**} \mathrm{U}(s) \mathbin{**} \mathrm{U}(\mathrm{U}(s)) \mathbin{**} \dots \mathbin{**} \mathrm{U}^{|s|}(s) \;\leq\; \mathrm{SqBd}(s \# s, s)$$

then g is polytime by limited recursion. Hence

$$\mathrm{tc}(s) \;:=\; \mathrm{sort}(g(s))$$

is polytime and computes the transitive closure of s. By construction $\mathrm{tc}(\cdot)$ is a course function.

We need a similar course-function for pairs of sequences.
Let $\mathrm{tc}_2(\langle s, t \rangle) = \langle c_1, \dots, c_k \rangle$ with $c_1 < \dots < c_k$ and

$$\{c_1, \dots, c_k\} = \{\langle d_i, e_j \rangle \;:\; 1 \leq i \leq m, 1 \leq j \leq n\}$$

[2]The *transitive closure* is generated using the obvious element relation on sequences which is given by a_i *is an element of* $\langle a_1, \dots, a_k \rangle$, $0 < i \leq k$.

where $\mathrm{tc}(s) = \langle d_1, \dots, d_m \rangle$ and $\mathrm{tc}(t) = \langle e_1, \dots, e_n \rangle$. The following equations are used to observe that $\mathrm{tc}_2(\cdot)$ is polytime. Let $s = \langle s_0, \dots, s_{k-1} \rangle$ and let $t = \langle t_0, \dots, t_{l-1} \rangle$.

$$\begin{aligned}
f(\langle s_0, \dots, s_{k-1} \rangle, a) &:= \langle \langle s_0, a \rangle, \dots, \langle s_{k-1}, a \rangle \rangle \leq \mathrm{SqBd}(s, s * a) \\
X(s, \langle t_0, \dots, t_{l-1} \rangle) &:= f(s, t_0) ** \dots ** f(s, t_{l-1}) \\
&\leq \mathrm{SqBd}(s \# t, s ** t) \\
\mathrm{tc}_2(\langle s, t \rangle) &:= \mathrm{sort}(X(\mathrm{tc}(s), \mathrm{tc}(t))).
\end{aligned}$$

By limited recursion both f and X are polytime. Thus, also $\mathrm{tc}_2(\cdot)$ is polytime. By construction $\mathrm{tc}_2(\cdot)$ is a course function.

We use $\mathrm{tc}_2(\cdot)$ to show that $\mathcal{E}$ and $\prec$ are polytime.

$$\begin{aligned}
\alpha \in \mathcal{E} &\iff \alpha = \check{2}^{\alpha_1} \check{+} \dots \check{+} \check{2}^{\alpha_k} \text{ with } \alpha_1, \dots, \alpha_k \in \mathcal{E} \text{ and } \\
&\qquad \alpha_k \prec \dots \prec \alpha_1. \\
\alpha \prec \beta &\iff \alpha, \beta \in \mathcal{E},\ \alpha = \check{2}^{\alpha_1} \check{+} \dots \check{+} \check{2}^{\alpha_k},\ \beta = \check{2}^{\beta_1} \check{+} \dots \check{+} \check{2}^{\beta_l} \\
&\qquad \text{and } \exists i{<}l\, (i \leq k \text{ and } \alpha_1 = \beta_1, \dots, \alpha_i = \beta_i \text{ and} \\
&\qquad (i = k \text{ or } \alpha_{i+1} \prec \beta_{i+1})).
\end{aligned}$$

We cannot apply Theorem 6.2.2 directly to this simultaneous definition because if we try to compute $\hat{0} \prec \check{2}^{\beta_1} \check{+} \check{2}^{\beta_2} =: \beta$ we need $\beta \in \mathcal{E}$ and for this $\beta_2 \prec \beta_1$. But $\langle \beta_2, \beta_1 \rangle$ does not occur in $\mathrm{tc}_2(\langle \hat{0}, \beta \rangle)$. Surely it is possible to change the definition of $\mathrm{tc}_2(\cdot)$ to overcome this lack, as $\langle \beta_2, \beta_1 \rangle < \langle \hat{0}, \beta \rangle$. But there is another possibility to show that $\mathcal{E}$ and $\prec$ are polytime which uses Theorem 6.2.2 and $\mathrm{tc}_2(\cdot)$. We define a more general relation $\prec'$. We obtain $\prec'$ by replacing $\mathcal{E}$ through `Seq` (the set of all Gödel numbers) in the definition of $\prec$. Let $\chi_{\prec'}$ be the characteristic function of $\prec'$, i.e.,

$$\chi_{\prec'}(\alpha, \beta) = \begin{cases} 1 : \alpha \prec' \beta \\ 0 : \textit{otherwise}, \end{cases}$$

and let $h(\langle \alpha, \beta \rangle) := \chi_{\prec'}(\alpha, \beta)$. Rewriting the definition of $\prec'$ we obtain a polytime function g satisfying

$$h(\langle \alpha, \beta \rangle) = g\Big(\langle \alpha, \beta \rangle, h^{\mathrm{tc}_2}(\langle \alpha, \beta \rangle)\Big) \leq 1,$$

therefore, Theorem 6.2.2 yields that h is polytime, thus also $\chi_{\prec'}$ and hence $\prec'$ are polytime. Now we define

$$\begin{aligned}\alpha \in \mathcal{E} \quad &\Longleftrightarrow \quad \texttt{Seq}(\alpha) \text{ and } \forall i < \mathrm{lh}(\alpha)\Big[\beta(i+1,\alpha) \in \mathcal{E} \quad \text{and} \\ &\qquad (i>0 \to \beta(i+1,\alpha) \prec' \beta(i,\alpha))\Big] \\ \alpha \prec \beta \quad &\Longleftrightarrow \quad \alpha \in \mathcal{E} \text{ and } \beta \in \mathcal{E} \text{ and } \alpha \prec' \beta.\end{aligned}$$

Using Theorem 6.2.2 with $\mathrm{tc}(\cdot)$ yields that $\mathcal{E}$ is polytime. Therefore, also $\prec$ is polytime.

Before we can define $\hat{+}$ on the exponential notations we need a successor function $\hat{\mathrm{S}}$ on them. To compute the successor on an exponential notation we need an auxiliary function F to manage carries. Therefore, we simultaneously define for $\alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k} \in \mathcal{E}$

$$\begin{aligned}F(\alpha) \;&:=\; \mu i \le k.\Big(i > 0 \text{ and } \forall j{<}k\,(j \ge i \to \alpha_j = \hat{\mathrm{S}}(\alpha_{j+1}))\Big) \\ \hat{\mathrm{S}}(\alpha) \;&:=\; \begin{cases} \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{i-1}}\check{+}\check{2}^{\hat{\mathrm{S}}\,\alpha_i} & : \alpha_k = \hat{0} \text{ and } i := F(\alpha) \\ \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}\check{+}\check{2}^{\hat{0}} & : \text{otherwise.}\end{cases}\end{aligned}$$

Clearly $F(\alpha) \le k = \mathrm{lh}(\alpha)$ and after proving $|\,\hat{\mathrm{S}}(\alpha)| \le |\alpha * \hat{0}|$ we can use Theorem 6.2.2 together with $\mathrm{tc}(\cdot)$ to see that both functions are polytime.

6.3.1 Lemma $|\,\hat{\mathrm{S}}(\alpha)| \le |\alpha * \hat{0}| \le |\alpha| + 4$.

Proof: Remember the definition

$$\begin{aligned} s * 0 \;&=\; (s0010)_2 \;=\; 16\cdot s + 2, \\ s * 1 \;&=\; (s0011)_2 \;=\; 16 \cdot s + 3, \\ s * (ai)_2 \;&=\; ((s*a)1i)_2 \;=\; 4\cdot(s*a) + 2 + i, \quad (i = 0,1 \text{ and } a \ne 0)\end{aligned}$$

and

$$\langle a_1,\ldots,a_k,a_{k+1}\rangle \;=\; \langle a_1,\ldots,a_k\rangle * a_{k+1}.$$

First we compute some constant notations and some binary lengths. Let $a = (a_1 \ldots a_k)_2$.

$$\begin{aligned} \hat{0} \;&=\; (0)_2 \;=\; 0 \\ \hat{1} \;&=\; (10)_2 \;=\; 2 \\ s \ne 0 \;\to\; |s*0| \;&=\; |\,(s0010)_2\,| \;=\; |s| + 4 \\ a \ne 0 \;\to\; |s*a| \;&=\; |\,(s001a_1 1a_2 \ldots 1a_k)_2\,| \;=\; |\,(s00)_2\,| + 2\cdot|a| \\ &=\; \begin{cases} 2\cdot|a| & : s = 0 \\ |s| + 2 + 2\cdot|a| & : s \ne 0.\end{cases}\end{aligned}$$

We start to prove the assertion by induction on $\alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k} = \langle\alpha_1,\ldots,\alpha_k\rangle$. If $k=0$, then $\alpha = \hat{0}$, hence $\hat{\mathrm{S}}(\hat{0}) = \check{2}^{\hat{0}} = \hat{0} * \hat{0}$. If $k>0$ and $\alpha_k \neq \hat{0}$, then $\hat{\mathrm{S}}(\alpha) = \langle\alpha_1,\ldots,\alpha_k,\hat{0}\rangle = \alpha * \hat{0}$. If $k>0$ and $\alpha_k = \hat{0}$, then let $i := F(\alpha)$. We have to distinguish the following cases. Let $\beta := \langle\alpha_1,\ldots,\alpha_{i-1}\rangle$.
If $i=k$ then we observe $\alpha = \beta * \hat{0}$ and

$$\hat{\mathrm{S}}(\alpha) \;=\; \beta * \hat{1} \;=\; \beta * (10)_2 \;=\; (\beta 001110)_2\,.$$

On the other hand we see

$$\alpha * \hat{0} \;=\; (\beta * \hat{0}) * \hat{0} \;=\; (\beta 00100010)_2 \;>\; \hat{\mathrm{S}}(\alpha).$$

If $i<k$ then we find $\alpha = \beta ** \langle\alpha_i,\ldots,\alpha_k\rangle$. Observe that $\Phi_{\mathcal{E}}(\alpha_j) = k-j$ for $j = i,\ldots,k$. Now the induction hypothesis produces

$$|\,\hat{\mathrm{S}}(\alpha_i)| \;\leq\; |\alpha_i * \hat{0}|. \tag{6.1}$$

This leads to

$$\begin{aligned}
|\,\hat{\mathrm{S}}(\alpha)| \;&=\; |\beta * \hat{\mathrm{S}}(\alpha_i)| \;=\; |\,(\beta 00)_2\,| + 2\cdot|\,\hat{\mathrm{S}}(\alpha_i)| \\
&\overset{(6.1)}{\leq}\; |\,(\beta 00)_2\,| + 2\cdot|\alpha_i * \hat{0}| \\
&=\; |\,(\beta 00)_2\,| + 2\cdot(|\alpha_i| + 4) \;=\; |\,(\beta 00)_2\,| + 2\cdot|\alpha_i| + 8
\end{aligned}$$

and

$$\begin{aligned}
|\alpha * \hat{0}| \;&=\; |(\beta ** \langle\alpha_i,\ldots,\alpha_k\rangle) * \hat{0}| \\
&\geq\; |\beta ** \langle\alpha_i,\hat{0},\hat{0}\rangle| \;=\; |((\beta * \alpha_i) * \hat{0}) * \hat{0}| \\
&=\; |\beta * \alpha_i| + 8 \;=\; |\,(\beta 00)_2\,| + 2\cdot|\alpha_i| + 8.
\end{aligned}$$

These two estimations together show $|\,\hat{\mathrm{S}}(\alpha)| \leq |\alpha * \hat{0}|$. □

We define the preaddition $\mathrm{pa}(\cdot,\cdot)$ which computes $\alpha \hat{+} \check{2}^{\beta}$ by

$$\mathrm{pa}(\alpha,\beta) \;:=\; \begin{cases}
\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}\check{+}\check{2}^{\beta} & : k=0 \text{ or } \beta \prec \alpha_k \\
\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{i-1}}\check{+}\check{2}^{\hat{\mathrm{S}}(\alpha_i)} & : \alpha_k = \beta \text{ and } i{:=}F(\alpha) \\
\mathrm{pa}\left(\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{k-1}},\beta\right) * \alpha_k & : \alpha_k \prec \beta
\end{cases}$$

where $\alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}$. In the next lemma we will see that $\mathrm{pa}(\cdot,\cdot)$ is polynomially bounded. Therefore, we can apply Theorem 6.2.2 together with the following polytime course function $\mathrm{initseq}(\cdot)$ to observe that $\mathrm{pa}(\cdot,\cdot)$ is polytime.

$$\mathrm{initseq}(\langle a_1,\ldots,a_k\rangle) \;:=\; \langle\langle a_1,\ldots,a_{k-1}\rangle,\ldots,\langle a_1\rangle,\langle\rangle\rangle.$$

6.3.2 Lemma $|\operatorname{pa}(\alpha,\beta)| \;\le\; |\alpha| + 2\cdot|\beta| + 8.$

Proof: We use induction on $\alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}$. If $k=0$ or $\beta \prec \alpha_k$, then

$$|\operatorname{pa}(\alpha,\beta)| \;=\; |\alpha * \beta| \;\le\; |\alpha| + 2 + 2\cdot|\beta| + 2.$$

If $\alpha_k = \beta$ then let $i := F(\alpha)$ and observe using $\gamma := \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{i-1}}$

$$\begin{aligned} |\operatorname{pa}(\alpha,\beta)| &= |\gamma * \hat{\mathrm{S}}(\alpha_i)| \;\le\; |(\gamma 00)_2| + 2\cdot(|\alpha_i| + 4) \\ &= |(\gamma 00)_2| + 2\cdot|\alpha_i| + 8 \;=\; |\gamma * \alpha_i| + 8 \\ &\le |\alpha| + 2\cdot|\beta| + 8. \end{aligned}$$

Otherwise, the induction hypothesis (i.h.) shows

$$\begin{aligned} |\operatorname{pa}(\alpha,\beta)| &= |\operatorname{pa}(\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{k-1}},\beta) * \alpha_k| \\ &= |\operatorname{pa}(\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{k-1}},\beta)| + 2 + 2\cdot\max(|\alpha_k|,1) \\ &\overset{i.h.}{\le} |\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_{k-1}}| + 2\cdot|\beta| + 8 + 2 + 2\cdot\max(|\alpha_k|,1) \\ &= |\alpha| + 2\cdot|\beta| + 8. \end{aligned}$$

□

Now we are able to define by limited recursion

$$\alpha\,\hat{+}(\check{2}^{\beta_1}\check{+}\ldots\check{+}\check{2}^{\beta_l}) \;:=\; \operatorname{pa}(\ldots\operatorname{pa}(\alpha,\beta_1)\ldots,\beta_l)$$

which is limited because

$$\begin{aligned} |\alpha\,\hat{+}\,\beta| &= |\operatorname{pa}(\ldots\operatorname{pa}(\alpha,\beta_1)\ldots,\beta_l)| \\ &\le |\alpha| + 2\cdot|\beta_1| + 8 + \ldots + 2\cdot|\beta_l| + 8 \\ &\le |\alpha| + |\beta| + 8\cdot l \;\le\; |\alpha| + 9\cdot|\beta|. \end{aligned}$$

Therefore, $\hat{+}$ is polytime.

Finally we want to observe that

$$\mathrm{T}_{\mathcal{E}}(n) \;=\; {\Phi_{\mathcal{E}}}^{-1}(n) \;=\; \text{"the unique } \alpha\in\mathcal{E} \text{ with } \Phi_{\mathcal{E}}(\alpha) = n\text{"}$$

is polytime. By limited recursion on $\alpha = \check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}$ we define

$$f(\alpha) \;:=\; \check{2}^{\hat{\mathrm{S}}(\alpha_1)}\check{+}\ldots\check{+}\check{2}^{\hat{\mathrm{S}}(\alpha_k)}$$

and compute

$$\begin{aligned} |f(\alpha)| &= 2\cdot|\hat{\mathrm{S}}(\alpha_1)| + 2 + \ldots + 2 + 2\cdot|\hat{\mathrm{S}}(\alpha_k)| \\ &\le 2\cdot(|\alpha_1| + 4) + 2 + \ldots + 2 + 2\cdot(|\alpha_k| + 4) \\ &= |\alpha| + 8\cdot k \;\le\; 9\cdot|\alpha|. \end{aligned}$$

f is polytime and it satisfies $\Phi_{\mathcal{E}}(f(\alpha)) = 2 \cdot \Phi_{\mathcal{E}}(\alpha)$. Using f we define, this time by limited recursion on notation,

$$\begin{array}{rcl} \mathrm{T}_{\mathcal{E}}(0) & := & \hat{0} \\ \mathrm{T}_{\mathcal{E}}(\,(ni)_2\,) & := & \begin{cases} f(\mathrm{T}_{\mathcal{E}}(n)) & : i = 0 \\ \hat{\mathrm{S}}(f(\mathrm{T}_{\mathcal{E}}(n))) & : i = 1. \end{cases} \end{array}$$

With the next lemma we obtain that $\mathrm{T}_{\mathcal{E}}$ is polytime.

6.3.3 Lemma $|\mathrm{T}_{\mathcal{E}}(n)| \leq 8 \cdot |n|^2$.

Proof: We use induction on n. If $n = 0$, then $|\mathrm{T}_{\mathcal{E}}(0)| = |\hat{0}| = 0 = 8 \cdot |0|^2$. If $n = 1$, then $|\mathrm{T}_{\mathcal{E}}(1)| = |\hat{1}| = |2| = 2 \leq 8 \cdot |1|^2$. For the induction step we consider $(ni)_2$ with $i = 0, 1$ and $n \geq 1$. We estimate

$$\begin{array}{rcl} |\mathrm{T}_{\mathcal{E}}(\,(ni)_2\,)| & \leq & |\hat{\mathrm{S}}(f(\mathrm{T}_{\mathcal{E}}(n)))| \;\leq\; |f(\mathrm{T}_{\mathcal{E}}(n))| + 4 \\ & \leq & |\mathrm{T}_{\mathcal{E}}(n)| + 8 \cdot \mathrm{lh}(\mathrm{T}_{\mathcal{E}}(n)) + 4 \\ & \leq & |\mathrm{T}_{\mathcal{E}}(n)| + 8 \cdot |n| + 4 \\ & \overset{i.h.}{\leq} & 8 \cdot |n|^2 + 8 \cdot |n| + 8 \;\leq\; 8 \cdot (|n| + 1)^2 \;=\; 8 \cdot |\,(ni)_2\,|^2. \end{array}$$

□

Altogether we have seen that the predicates $\mathcal{E}, \prec$ and the functions $\hat{+}, \hat{2}$ and $\mathrm{T}_{\mathcal{E}}$ are polytime.

Finally we prove that the predecessor function on the exponential notations

$$\hat{\mathrm{P}}(\alpha) := \begin{cases} \hat{0} : \alpha = \hat{0} \\ \beta : \text{for that } \beta \text{ with } \beta \hat{+} \hat{1} = \alpha \end{cases}$$

is not a polytime function. We can show

6.3.4 Theorem $\hat{\mathrm{P}}$ *is not polynomially bounded.*

Proof: Obviously $|\,\hat{2}^{\mathrm{T}_{\mathcal{E}}(n)}\,| > 1$ for $n > 0$, hence

$$\begin{array}{rcl} |\,\hat{\mathrm{P}}(\hat{2}^{\mathrm{T}_{\mathcal{E}}(n)})| & = & |\mathrm{T}_{\mathcal{E}}(2^n - 1)| \\ & = & |\mathrm{T}_{\mathcal{E}}(2^{n-1} + \ldots + 2^0)| \\ & \geq & 2 \cdot n \geq 2^{|n|}. \end{array}$$

On the other hand we compute $|\,\hat{2}^{\mathrm{T}_{\mathcal{E}}(n)}\,| = 2 \cdot |\mathrm{T}_{\mathcal{E}}(n)| \leq 16 \cdot |n|^2$. If $\hat{\mathrm{P}}$ would be polynomially bounded then there has to be some monotone polynomial $p(x)$ with $|\,\hat{\mathrm{P}}(x)| \leq p(|x|)$. But then

$$2^{|n|} \leq |\,\hat{\mathrm{P}}(\hat{2}^{\mathrm{T}_{\mathcal{E}}(n)})| \leq p(|\,\hat{2}^{\mathrm{T}_{\mathcal{E}}(n)}\,|) \leq p(16 \cdot |n|^2)$$

which yields a contradiction for large n. □

Chapter 7

Bounded Predicative Arithmetic (BPA)

In the introduction we motivated that if the aspired Dynamic Ordinal $\Phi_{\mathcal{E}}(\alpha(x))$ of a theory is not close enough to x (i.e., eventually $\Phi_{\mathcal{E}}(\alpha(x)) \geq 2^x$) then we have to assume the existence of a value a which bounds all exponential notations below $\alpha(x)$. This value is not allowed to bound the length of an induction – otherwise this would influence the Dynamic Ordinal in a way that a in general cannot bound all exponential notations below this Dynamic Ordinal. Thus, from the point of view of induction, a has to be impredicative.

In [5] Bellantoni and Cook made observations which are related to this. They presented a new recursion theoretic characterization of the polytime functions and considered functions with two kinds of arguments:

$$f(\vec{x};\vec{a})$$

$$\nearrow \quad \nwarrow$$

normal safe

The difference between the two sorts of arguments is that a value in the normal position can be used to do binary recursion up to that value. In the safe position you are "safe" to use impredicative values which come along as the intermediate values of a recursion.

We will capture in the formulation of bounded predicative arithmetic the idea, that the individuals are divided into a predicative and an impredicative part at which only the predicative values are allowed to bound induction.

7.1 The language

The individual universe I of a structure $\mathcal{S}$ for *bounded predicative arithmetic* contains a sub-universe I_p of predicative values, i.e., $\emptyset \neq I_p \subseteq I$. I_p is closed under the polytime functions from the finite set $\mathcal{F}^p$ which is given by

$$\{0, \mathrm{S}, +, \cdot, |x|, \lfloor \tfrac{1}{2}x \rfloor, x \# y, x \dot{-} y, \mathrm{MSP}, \mathrm{LSP}\}$$

and under the polytime functions from the finite set $\mathcal{F}^i$ which is given by

$$\begin{aligned}\{&\mathrm{S}, \mathrm{S}_0, \mathrm{S}_1,\\ &*, **, \mathrm{first}, \mathrm{last}, \mathrm{trunc_l}, \mathrm{trunc_r}, \beta, \mathrm{lh},\\ &\qquad \hat{0}, \hat{1}, x \check{+} \check{2}^y, \hat{2}^x, \hat{+}, \widehat{2\cdot}\, x, \mathrm{T}_{\mathcal{E}}(x)\}.\end{aligned}$$

Furthermore, I_p is closed under some weak form of induction.

Not much structure is assumed for the impredicative part of the individual universe I. Only some arithmetical connections are given on the impredicative part between the graphs of the polytime functions from the set $\mathcal{F}^i$.

Notice: We do not assume that any function is total on I.

Keeping this picture in mind we define a formal language $\mathcal{L}_{BPA}$ which is a TAIT-style language of first order logic with equality containing

- two sorts of individual variables: *predicative* ones which are denoted by $x_0, x_1, \ldots, x, y, z, \ldots$, and *impredicative* ones which are denoted by $a_0, a_1, \ldots, a, b, c, \ldots$. Thus, an assignment Φ for $\mathcal{S}$ of the variables satisfies $\Phi(x) \in I_p$ and $\Phi(a) \in I$. Actually we think of *four* sorts of individual variables: two for free and two for bounded variables - but we will not use a special TEX-Font (like Gothic or perhaps Klingonic) to distinguish between the free and bounded ones.

- logical symbols $=, \neq, \wedge, \vee, \forall, \exists$

- nonlogical symbols:

  - a function symbol $\underline{\mathrm{f}}$ with arity $\mathrm{ar}(f)$ for every polytime function $f \in \mathcal{F}^p \cup \mathcal{F}^i$. We think of $\underline{\mathrm{f}}$ living in the predicative universe, i.e., $\underline{\mathrm{f}}^{\mathcal{S}} \colon I_p^{\mathrm{ar}(f)} \to I_p$. Formally this will be expressed in the definition of terms.

– for each $\mathrm{ar}(f)$-ary $f \in \mathcal{F}^i$ two $(\mathrm{ar}(f)+1)$-ary predicate symbols, $\mathcal{G}_f$ for the graph of f and $\mathcal{G}_f^c$ for its complement. These predicates speak about the whole universe I, i.e., ${\mathcal{G}_f}^{\mathcal{S}} \subseteq I^{\mathrm{ar}(f)+1}$, $\mathcal{G}_f^{c\mathcal{S}} = I^{\mathrm{ar}(f)+1} \setminus {\mathcal{G}_f}^{\mathcal{S}}$.

– for each $\mathrm{ar}(P)$-ary $P \in \mathcal{P}^i := \{\leq, \mathrm{Bit}, \mathtt{Seq}, \mathcal{E}, \prec\}$ two predicate symbols of arity $\mathrm{ar}(P)$, $\underline{\mathrm{P}}$ for P and $\underline{\mathrm{P}^{\mathrm{c}}}$ for its complement. Again these predicates speak about the whole universe I, i.e., $\underline{\mathrm{P}}^{\mathcal{S}} \subseteq I^{\mathrm{ar}(P)}$, $\underline{\mathrm{P}^{\mathrm{c}}}^{\mathcal{S}} = I^{\mathrm{ar}(P)} \setminus \underline{\mathrm{P}}^{\mathcal{S}}$.

- auxiliary symbols (,).

Sometimes we want to have an extended language $\mathcal{L}_{BPA}(\mathcal{X})$ containing additional set variables, denoted by $X_0, X_1, \ldots, Y, Z, \ldots$, and the binary predicates $\in, \notin$ as logical symbols. Then $\Phi(X) \subseteq I$ and $\in^{\mathcal{S}}$ is the usual "element"-relation.

In the sequel we will write $\leq$ and $\not\leq$ instead of $\underline{\leq}$ resp. $\underline{\leq^{\mathrm{c}}}$. For the rest of this chapter we fix $\mathcal{L}_{BPA}$ resp. $\mathcal{L}_{BPA}(\mathcal{X})$ as the underlying formal language. It will be clear from the context which of both is considered.

The predicative variables are often called *normal*, the impredicative ones *safe*. We use φ as an individual variable if we do not care about its sort.

The normal variables range over I_p and the safe variables over I. We inductively define the *predicative* or *normal terms* respecting these different meanings by:

1. Normal variables are normal terms

2. If $\underline{\mathrm{f}}$ is a sign for a polytime function $f \in \mathcal{F}^p \cup \mathcal{F}^i$ and $t_1, \ldots, t_n$ are normal terms, then $(\underline{\mathrm{f}}\, t_1 \ldots t_n)$ is a normal term.

A *term* is a normal term or a safe variable.

As the predicates are intended to speak about the whole universe I we can define *formulas* in the usual way starting from the atomic formulas using the terms[1]. The characteristic feature of a Tait-style language is that negation is not a logical symbol but can be defined as a syntactic operation $\neg$ according to the de Morgan-laws[2]. With

[1] Cf. Chapter 3.

[2] Cf. Chapter 3.

$\mathsf{FV}(F)$ we denote the set of all free variables, with $\mathsf{nFV}(F)$ that of all normal and with $\mathsf{sFV}(F)$ that of all safe variables that occur in F. We use $<$, $\rightarrow$, $\leftrightarrow$, etc. as defined symbols[3]. A term or formula which contains no normal variables is called *predicative ground*.

Notice: The only predicative ground terms are the ground $\mathcal{L}_{BPA}$-terms and the impredicative variables.

We can interpret each term t in $\mathcal{S}$ under Φ. This yields a value $t^{\mathcal{S}}[\Phi] \in I$. Furthermore, if t is a normal term then $t^{\mathcal{S}}[\Phi] \in I_p$. Thus, also the $\mathcal{L}_{BPA}$-formulas are interpretable in $\mathcal{S}$ under Φ. With $\mathbb{N}$ we mean the standard model of the natural numbers which consists of the identical predicative and impredicative part ω. The interpretations of the other non-logical symbols are given in Appendix A. We shortly write $\mathbb{N} \models F_{\varphi}[n]$ instead of $\mathbb{N} \models F[\Phi]$ for some Φ with $\Phi(\varphi) = n$ if $\mathsf{FV}(F) \subset \{\varphi\}$.

Bounded quantifiers and bounded formulas play an important role in bounded arithmetics. We abbreviate

$$\begin{array}{lcl}
\forall\varphi{\leq}u\, A(\varphi) & :\equiv & \forall\varphi\,[\varphi \leq u \rightarrow A(\varphi)] \\
\exists\varphi{\leq}u\, A(\varphi) & :\equiv & \exists\varphi\,[\varphi \leq u \wedge A(\varphi)] \\
\forall\varphi{<}u\, A(\varphi) & :\equiv & \forall\varphi{\leq}u\,[\varphi < u \rightarrow A(\varphi)] \\
\exists\varphi{<}u\, A(\varphi) & :\equiv & \exists\varphi{\leq}u\,[\varphi < u \wedge A(\varphi)]
\end{array}$$

and call these quantifiers *bounded*. A formula containing only bounded quantifiers is called a *bounded formula*. We call a bounded quantifier *normal* if the bounding term u is normal. We call a normal bounded quantifier *sharply bounded* if the bounding term u is of the shape $|u'|$. We call a bounded formula *normal* (resp. *sharply bounded*) if all quantifiers occurring in it are normal (resp. sharply bounded).

7.2 The theories

We define the set of *predicative bounded formulas* PBF as the set of all bounded $\mathcal{L}_{BPA}$-formulas whose quantifiers respect the ontological meaning of normal and safe variables. E.g., a normal variable bounded by a safe variable yields in some sense an unbounded quantifier over the predicative part.

[3] Cf. Chapter 3.

7.2.1 Definition PBF *is inductively defined by the following clauses.*

1. *All atomic* $\mathcal{L}_{BPA}$*-formulas are in* PBF*.*

2. PBF *is closed under* $\wedge, \vee$*.*

3. *If* $A \in$ PBF*,* x *is a normal variable and* t *is a normal term, then* $\exists x{\le}t\, A$ *and* $\forall x{\le}t\, A$ *are in* PBF*.*

4. *If* $A \in$ PBF*,* a *is a safe variable and* s *is a term, then* $\exists a{\le}s\, A$ *and* $\forall a{\le}s\, A$ *are in* PBF*.*

Let $\mathrm{PBF}(\mathcal{X})$ *be the obvious extension of this definition to* $\mathcal{L}_{BPA}(\mathcal{X})$*-formulas.*

In PBF we distinguish special sets of formulas ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n, {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n$ and ${}^{\mathrm{p}}\Delta^{\mathrm{b}}_0$.

7.2.2 Definition 1. ${}^{\mathrm{p}}\Delta^{\mathrm{b}}_0 = {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_0 = {}^{\mathrm{p}}\Pi^{\mathrm{b}}_0$ *is the set consisting of all sharply bounded* PBF*-formulas.*

2. ${}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}$ *is the set of* PBF*-formulas of the form*

$$\forall a_1{\le}s_1 \ldots \forall a_p{\le}s_p\, \forall x{\le}t\, A(a_1, \ldots, a_p, x)$$

for some terms $s_1, \ldots, s_p, t$ *and* $A \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$*.*

3. ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}$ *is the set of* PBF*-formulas of the form*

$$\exists a_1{\le}s_1 \ldots \exists a_p{\le}s_p\, \exists x{\le}t\, A(a_1, \ldots, a_p, x)$$

for some terms $s_1, \ldots, s_p, t$ *and* $A \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n$*.*

Let ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_\infty := \bigcup_{n\in\omega} {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$ *and* ${}^{\mathrm{p}}\Pi^{\mathrm{b}}_\infty := \bigcup_{n\in\omega} {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n$*. Again let* ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})$*,* ${}^{\mathrm{p}}\Pi^{\mathrm{b}}_n(\mathcal{X})$*,* ${}^{\mathrm{p}}\Delta^{\mathrm{b}}_0(\mathcal{X})$ *be the obvious relativization of this definition to* $\mathcal{L}_{BPA}(\mathcal{X})$*-formulas.*

Notice: Formulas from ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$ etc. are always strict.

In order to develop the relevant theories we first state some axioms. Let ${}^{\mathrm{p}}$BASIC be a finite set of defining axioms for the non-logical symbols of $\mathcal{L}_{BPA}$ (an axiom will be a propositional combination of atomic formulas), i.e.

- axioms for the functions in $\mathcal{F}^p$ like the set BASIC from [6] plus some more. These axioms are formulated with normal variables as the functions live in the predicative universe.

- axioms for the predicates in $\mathcal{P}^i$ and for the graphs of functions in $\mathcal{F}^i$ (such an axiomatization is given in Appendix B)
- for each $f \in \mathcal{F}^i$ axioms of the form
  - $\mathcal{G}_f(\vec{x}, \underline{\mathrm{f}}\,\vec{x})$
  - $\mathcal{G}_f(\vec{a}, b) \wedge \mathcal{G}_f(\vec{a}, c) \rightarrow b = c.$

Beside this we need several *induction axioms*. Let

$$\begin{aligned} Ind(F,y,x) &\;:\equiv\; F_y(0) \wedge \forall y{<}x\,(F \rightarrow F_y(\mathrm{S}\,y)) \rightarrow F_y(x), \\ PInd(F,y,x) &\;:\equiv\; F_y(0) \wedge \forall y{\leq}x\,(F_y(\lfloor \tfrac{1}{2} y \rfloor) \rightarrow F) \rightarrow F_y(x). \end{aligned}$$

Let $|x|_0 :\equiv x$ and $|x|_{m+1} :\equiv |(|x|_m)|$. We obtain several axiom schemas for $\Psi \subset \mathrm{PBF}$:

$$\begin{aligned} \Psi\text{-Ind} &:= \{Ind(F,y,x) : F \in \Psi\} \\ \Psi\text{-LInd} &:= \{Ind(F,y,|x|) : F \in \Psi\} \\ \Psi\text{-LLInd} &:= \{Ind(F,y,||x||) : F \in \Psi\} \\ \Psi\text{-L}^m\text{Ind} &:= \{Ind(F,y,|x|_m) : F \in \Psi\} \\ \Psi\text{-PInd} &:= \{PInd(F,y,x) : F \in \Psi\} \\ \Psi\text{-PLInd} &:= \{PInd(F,y,|x|) : F \in \Psi\} \\ \Psi\text{-PL}^m\text{Ind} &:= \{PInd(F,y,|x|_m) : F \in \Psi\} \end{aligned}$$

In particular the following theories will be of interest:

$$\begin{aligned} {}^{\mathrm{p}}\mathrm{R}^n_2 &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-LLInd} \\ {}^{\mathrm{p}}\mathrm{S}^n_2 &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-LInd} \\ {}^{\mathrm{p}}\mathrm{T}^n_2 &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-Ind} \\ {}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X}) &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-LLInd} \\ {}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X}) &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-LInd} \\ {}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X}) &:= {}^{\mathrm{p}}\mathrm{BASIC} + {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-Ind.} \end{aligned}$$

Let $\mathcal{T}$ be a theory ${}^{\mathrm{p}}$BASIC+ some *induction schema*, then this *induction schema* is also written as $\mathcal{T}$-Ind. We will often omit to mention ${}^{\mathrm{p}}$BASIC when we speak about theories, e.g., we will say Φ-L^{m}Ind instead of ${}^{\mathrm{p}}\mathrm{BASIC} + \Phi\text{-L}^m\mathrm{Ind}$.
Notice: We have $Ind(F,x,t) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_\infty$ for $F \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_\infty$ and normal t, and

$Ind(F, x, t) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}(\mathcal{X})$ for $F \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}(\mathcal{X})$ and normal t. The same holds for $PInd$.

With an *instance of a formula* F we mean a formula which results from replacing all variables of F by some terms respecting the sort (normal, safe) of the variable: normal variables are replaced by normal terms and safe variables by arbitrary terms.

7.2.3 Definition *We inductively define $\mathcal{T} \vdash \Delta$ for finite sets of formulas Δ in the language of $\mathcal{T}$ by the following clauses.*

(AxL) *$\mathcal{T} \vdash \Delta$ holds if Δ contains a logical axiom $\neg A, A$ for some atomic formula A.*

(AxE) *$\mathcal{T} \vdash \Delta$ holds if Δ contains an equality axiom of the form $(s = s)$ or $(s = t \wedge A(s) \rightarrow A(t))$ for some atomic formula $A(\varphi)$ and terms s, t.*

(Ax^{p}B) *$\mathcal{T} \vdash \Delta$ holds if Δ contains an instance of an axiom from* ${}^{\mathrm{p}}$BASIC.

($\mathcal{T}$-IND) *$\mathcal{T} \vdash \Delta$ holds if Δ contains an instance of a formula from* $\mathcal{T}$-Ind.

($\wedge$) *$\mathcal{T} \vdash \Delta, F_0 \wedge F_1$ holds if $\mathcal{T} \vdash \Delta, F_i$ for all $i \in \{0, 1\}$.*

($\vee$) *$\mathcal{T} \vdash \Delta, F_0 \vee F_1$ holds if $\mathcal{T} \vdash \Delta, F_i$ for some $i \in \{0, 1\}$.*

($\forall$) *$\mathcal{T} \vdash \Delta, \forall\varphi\, F$ holds if there is some ψ not occurring in $\Delta, \forall\varphi\, F$ with [φ safe $\implies$ ψ safe] and $\mathcal{T} \vdash \Delta, F_{\varphi}(\psi)$.*

($\exists$) *$\mathcal{T} \vdash \Delta, \exists\varphi\, F$ holds if there is some term s with [φ normal $\implies$ s normal] and $\mathcal{T} \vdash \Delta, F_{\varphi}(s)$.*

(Cut) *$\mathcal{T} \vdash \Delta$ holds if there is some formula F with $\mathcal{T} \vdash \Delta, F$ and $\mathcal{T} \vdash \Delta, \neg F$.*

7.2.4 Remark *The introduced formal derivation systems are complete with respect to the BPA-models of the universal closure of the underlying theory. I.e., let T be the set of all nonlogical axioms occuring in the previous definition,*

$$T = \{F : F \text{ is an instance of a formula from } {}^{\mathrm{p}}\mathrm{BASIC} \cup \mathcal{T}\text{-Ind}\},$$

then the truth of F in all BPA-models of T implies $\mathcal{T} \vdash F$.

The introduced systems allow partial cut-elimination[4], i.e., the cuts can be reduced to formulas of the complexity of the axioms which are ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}(\mathcal{X})$-formulas.

Furthermore, we obtain a normal form for derivations. Let $\mathcal{T}$ be a theory formulated in $\mathcal{L}_{BPA}$ and Δ a finite set of $\mathcal{L}_{BPA}$-formulas. Then we can show that $\mathcal{T} \vdash \Delta$ iff Δ is derivable in the restriction of the calculus defined in 7.2.3, in which the cut-formulas are restricted to ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}$-formulas and only $(\forall)$-inferences eliminate variables. The last–metioned means if $\Delta_i, i{\le}k \implies \Gamma$ is an inference according to $(\wedge)$, $(\vee)$, $(\exists)$, (Cut), then $\mathsf{FV}(\Delta_i) \subset \mathsf{FV}(\Gamma)$ for $i \le k$, and if $\Gamma, F_\varphi(\psi) \implies \Gamma, \forall\varphi\, F$ is an inference according to $(\forall)$ then $\mathsf{FV}(\Gamma, F_\varphi(\psi)) \subset \mathsf{FV}(\Gamma, \forall\varphi\, F) \cup \{\psi\}$. This eliminated variable has to be the eigenvariable of the inference. We call such a restricted derivation a *normal derivation*. In the following we only consider normal derivations without particularly mentioning it (i.e., we write $\mathcal{T} \vdash \Delta$ and we mean that Δ is derivable with a normal derivation).

This normal form is somehow part of the normal form which Buss et al. call "a bounded proof which has no free cuts, is in free variable normal form and is restricted by parameter variables" (cf. [6], p. 77, Theorem 9). In essential the normal form defined here is that part of the latter normal form which is needed for the forthcoming.

With help of the following lemma we also obtain normal derivations of $\mathcal{L}_{BPA}(\mathcal{X})$-formulas for a theory formulated in $\mathcal{L}_{BPA}(\mathcal{X})$: no set variable disappears by an application of an inference. Let $F_X(\{a : G(a)\})$ – or shortly $F_X(G(.))$ – be the expression obtained from F by replacing all occurrences of $s \in X$ by $G(s)$ and all occurrences of $s \notin X$ by $\neg G(s)$. If F and G are $\mathcal{L}_{BPA}(\mathcal{X})$-formulas so is $F_X(\{a : G(a)\})$.

7.2.5 Lemma *Let F be an $\mathcal{L}_{BPA}(\mathcal{X})$-formula and $G(a) \in {}^{\mathrm{p}}\Delta^{\mathrm{b}}_0(\mathcal{X})$.*

$$\mathcal{T}(\mathcal{X}) \vdash F \implies \mathcal{T}(\mathcal{X}) \vdash F_X(\{a : G(a)\}).$$

Proof by induction on the derivation $\mathcal{T}(\mathcal{X}) \vdash F$: For the only critical case, the $\mathcal{T}(\mathcal{X})$-Ind axioms, we observe

$$F \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_k(\mathcal{X}) \implies F_X(\{a : G(a)\}) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_k(\mathcal{X}).$$

[4]Cf. [2].

□

We give two consequences of normal derivability.

7.2.6 Theorem *Let F be an $\mathcal{L}_{BPA}$-formula.*

$$\mathcal{T}(\mathcal{X}) \vdash F \implies \mathcal{T} \vdash F.$$

Proof: All formulas which occur in a normal derivation of an $\mathcal{L}_{BPA}$-formula are in $\mathcal{L}_{BPA}$. □

7.2.7 Remark *All formulas which occur in a normal derivation $T \vdash F$ of $F \in \mathrm{PBF}$ (resp. $F \in \mathrm{PBF}(\mathcal{X})$) are in* PBF *(resp. in* $\mathrm{PBF}(\mathcal{X})$*).*

Chapter 8

Bounded Arithmetic (BA) and BPA

In this chapter we investigate the relationship between the introduced theories ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{R}^n_2$, ${}^{\mathrm{p}}\mathrm{S}^n_2$, ${}^{\mathrm{p}}\mathrm{T}^n_2$, ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X})$ and the usual considered theories of Bounded Arithmetic $\mathrm{s}\Sigma^{\mathrm{b}}_n$-$\mathrm{L}^m\mathrm{Ind}$, sR^n_2, S^n_2, T^n_2, $\mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$, $\mathrm{sR}^n_2(\mathcal{X})$, $\mathrm{S}^n_2(\mathcal{X})$, $\mathrm{T}^n_2(\mathcal{X})$. For theories T_1, T_2 formulated in the same language $T_1 \subset T_2$ will denote that T_2 is an extension of T_1 and $T_1 = T_2$ will denote $T_1 \subset T_2$ and $T_2 \subset T_1$.

8.1 Fragments of BA

Let $\mathcal{L}_{BA}$ (resp. $\mathcal{L}_{BA}(\mathcal{X})$) be the sublanguage of $\mathcal{L}_{BPA}$ (resp. $\mathcal{L}_{BPA}(\mathcal{X})$) consisting of the predicative part of $\mathcal{L}_{BPA}$ (resp. $\mathcal{L}_{BPA}(\mathcal{X})$), i.e., the variables $x_0, x_1, \dots,\ x, y, z, \dots,\ =, \neq, \wedge, \vee, \forall, \exists$ and the function symbols $\underline{\mathrm{f}}$ for each $f \in \mathcal{F}^p \cup \mathcal{F}^i$ (resp. set variables $X_0, X_1, \dots, Y, Z, \dots,$ and the binary predicates $\in, \notin$). We consider $\mathcal{G}_f$,$\mathcal{G}^c_f$ as defined symbols via $\mathcal{G}_f(\vec{x}, y) :\equiv (\underline{\mathrm{f}}\,\vec{x} = y)$ and $\mathcal{G}^c_f(\vec{x}, y) :\equiv (\underline{\mathrm{f}}\,\vec{x} \neq y)$.

Our definition of $\mathcal{L}_{BA}$ (and therefore also of Σ^{b}_n, S^n_2, etc.) differs from those occuring in the literature (cf. [6, 15]) in so far that we consider an extension of the original theory S^1_2 by definitions. Here $\mathcal{L}_{BA}$ contains finitely many additional function symbols for polytime functions. But, as Buss has shown in [6], every polytime function is Σ^{b}_1-definable in S^1_2 and, therefore, can be used in an extension by definitions, and also in the induction axioms.

8.1.1 Definition *1.* $\Delta^{\mathrm{b}}_0 = \Sigma^{\mathrm{b}}_0 = \Pi^{\mathrm{b}}_0$ *is the set of sharply bounded $\mathcal{L}_{BA}$-formulas.*

2. Π^{b}_{n+1} *is the smallest set of $\mathcal{L}_{BA}$-formulas which contains Σ^{b}_n and is closed under $\wedge$, $\vee$, $(\forall x{\le}t\,)$ and $(\exists x{\le}|t|\,)$.*

3. $\Sigma^{\mathrm{b}}_{n+1}$ *is the smallest set of $\mathcal{L}_{BA}$-formulas which contains Π^{b}_n and is closed under $\wedge$, $\vee$, $(\exists x{\le}t\,)$ and $(\forall x{\le}|t|\,)$.*

We obtain $\Sigma^{\mathrm{b}}_n(\mathcal{X})$, $\Pi^{\mathrm{b}}_n(\mathcal{X})$, $\Delta^{\mathrm{b}}_0(\mathcal{X})$ by relativizing this definition to $\mathcal{L}_{BA}(\mathcal{X})$. Next we define *prenex* or *strict* versions of the former sets.

8.1.2 Definition *1.* $\mathrm{s}\Delta^{\mathrm{b}}_0 = \mathrm{s}\Sigma^{\mathrm{b}}_0 = \mathrm{s}\Pi^{\mathrm{b}}_0$ *is the set of sharply bounded $\mathcal{L}_{BA}$-formulas.*

2. $\mathrm{s}\Pi^{\mathrm{b}}_{n+1}$ *is the set of $\mathcal{L}_{BA}$-formulas of the form*

$$\forall x{\le}t\, A(x)$$

for some term t and $A \in \mathrm{s}\Sigma^{\mathrm{b}}_n$.

3. $\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}$ *is the set of $\mathcal{L}_{BA}$-formulas of the form*

$$\exists x{\le}t\, A(x)$$

for some term t and $A \in \mathrm{s}\Pi^{\mathrm{b}}_n$.

We obtain $\mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})$, $\mathrm{s}\Pi^{\mathrm{b}}_n(\mathcal{X})$, $\mathrm{s}\Delta^{\mathrm{b}}_0(\mathcal{X})$ by relativizing this definition to $\mathcal{L}_{BA}(\mathcal{X})$.
Notice: $\mathrm{s}\Sigma^{\mathrm{b}}_n = \{F \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n : \text{ only normal variables occur in } F\}$.
Similarly for $\mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})$, $\mathrm{s}\Pi^{\mathrm{b}}_n$ etc.

Let BASIC be the set ${}^{\mathrm{p}}$BASIC restricted to $\mathcal{L}_{BA}$ (and interpreting $\mathcal{G}_f$ and $\mathcal{G}^c_f$ in the way described above). The following theories have special names:

$$\begin{aligned}
\mathrm{sR}^n_2 &= \mathrm{BASIC} + \mathrm{s}\Sigma^{\mathrm{b}}_n\text{-LLInd} \\
\mathrm{R}^n_2 &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n\text{-LLInd} \\
\mathrm{S}^n_2 &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n\text{-LInd} \\
\mathrm{T}^n_2 &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n\text{-Ind} \\
\mathrm{sR}^n_2(\mathcal{X}) &= \mathrm{BASIC} + \mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-LLInd} \\
\mathrm{R}^n_2(\mathcal{X}) &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-LLInd} \\
\mathrm{S}^n_2(\mathcal{X}) &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-LInd} \\
\mathrm{T}^n_2(\mathcal{X}) &= \mathrm{BASIC} + \Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-Ind}
\end{aligned}$$

Speaking about theories we usually omit BASIC, e.g., we say Φ-L^{m}Ind and mean BASIC + Φ-L^{m}Ind.

Let the *sharply bounded collection axioms* be defined by

$$BB(F, y_0, y_1, x_0, x_1) \equiv (\forall y_0 \le |x_0|)(\exists y_1 \le x_1)F$$
$$\rightarrow (\exists w \le \mathrm{SqBd}(x_1, x_0))(\forall y_0 \le |x_0|)F_{y_1}(\beta(\mathrm{S}\, y_0, w)).$$

The associated schema is denoted by BB Ψ for sets of formulas Ψ.

8.1.3 Lemma *1.* BB sΣ^{b}_n $\vdash$ "$\Sigma^{\mathrm{b}}_n = \mathrm{s}\Sigma^{\mathrm{b}}_n$"

2. BB sΣ^{b}_n $\vdash$ "$\Pi^{\mathrm{b}}_n = \mathrm{s}\Pi^{\mathrm{b}}_n$" □

8.2 Comparing theories of BA

We summarize the connections between the different axioms.

8.2.1 Lemma BASIC *proves*

1. $Ind(\neg F_y(x \dot{-} y), y, x) \rightarrow Ind(F, y, x)$
2. $PInd(F_y(|y|), y, x) \rightarrow LInd(F, y, x)$
3. $LInd(F_y(\mathrm{MSP}(x, |x| \dot{-} y)), y, x) \rightarrow PInd(F, y, x)$
4. $PInd((\forall y \le x)(\forall u \le x)(u \le z+1 \wedge y+u \le x \wedge F \rightarrow F_y(y+u)), z, x)$
   $\rightarrow Ind(F, y, x)$
5. $LInd(G, z, x_0) \rightarrow BB(F, y_0, y_1, x_0, x_1)$ *for*
   $G :\equiv (\exists w \le \mathrm{SqBd}(x_1, x_0))(\forall y_0 \le |x_0|)(y_0 \le z \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w)))$
6. $PLInd([(\forall y_2 \le |x_0|)(\exists w \le \mathrm{SqBd}(x_1, x_0))(\forall y_0 \le |x_0|)$
   $(y_2 \le y_0 \le y_2 + z \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w)))], z, x_0) \rightarrow BB(F, y_0, y_1, x_0, x_1)$

Proof: Proofs of 1.-5. can be found in [6]. For 6. let $G(z, x_0, x_1)$ be the formula

$$(\forall y_2 \le |x_0|)(\exists w \le \mathrm{SqBd}(x_1, x_0))$$
$$(\forall y_0 \le |x_0|)(y_2 \le y_0 \le y_2 + z \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w))).$$

Assume $(\forall y_0 \le |x_0|)(\exists y_1 \le x_1)F$, then we have

$$(\forall y_0 \le |x_0|)(\exists w \le \mathrm{SqBd}(x_1, x_0))F_{y_1}(\beta(\mathrm{S}\, y_0, w)),$$

hence $G(0, x_0, x_1)$.

Now assume $G(\lfloor \frac{1}{2}z \rfloor, x_0, x_1)$ for some $z \leq |x_0|$. Let $y_2 \leq |x_0|$. Then there are $w_1, w_2 \leq \mathrm{SqBd}(x_1, x_0)$ with

$$(\forall y_0 {\leq} |x_0|\,)(y_2 \leq y_0 \leq y_2 + \lfloor \tfrac{1}{2}z \rfloor \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w_1)))$$

and

$$\begin{aligned} &y_2 + 1 + \lfloor \tfrac{1}{2}z \rfloor \leq |x_0| \rightarrow (\forall y_0 {\leq} |x_0|\,) \\ &\quad [y_2 + 1 + \lfloor \tfrac{1}{2}z \rfloor \leq y_0 \leq y_2 + 1 + 2 \cdot \lfloor \tfrac{1}{2}z \rfloor \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w_2))]. \end{aligned}$$

Let w be

$$\langle w_{1,0}, \ldots, w_{1,y_2+\lfloor \frac{1}{2}z \rfloor}, w_{2,y_2+\lfloor \frac{1}{2}z \rfloor+1}, \ldots, w_{2,y_2+2\cdot\lfloor \frac{1}{2}z \rfloor+1} \rangle$$

where $w_{i,j} := \beta(j+1, w_i)$, then by construction

$$(\forall y_0 {\leq} |x_0|\,)(y_2 \leq y_0 \leq y_2 + z \rightarrow F_{y_1}(\beta(\mathrm{S}\, y_0, w))),$$

hence $G(z, x_0, x_1)$. Therefore, we obtain $G(|x_0|, x_0, x_1)$ by applying $PLInd(G(z, x_0, x_1), z, x_0)$, thus – choosing $y_2 := 0$ –

$$(\exists w {\leq} \mathrm{SqBd}(x_1, x_0)\,)(\forall y_0 {\leq} |x_0|\,) F_{y_1}(\beta(\mathrm{S}\, y_0, w)).$$

□

8.2.2 Lemma *Let $m, n \geq 0$. Let Φ be one of $\mathrm{s}\Sigma^{\mathrm{b}}_n, \mathrm{s}\Pi^{\mathrm{b}}_n, \Sigma^{\mathrm{b}}_n, \Pi^{\mathrm{b}}_n$ and $\neg\Phi$ its dual class.*

1. $\Phi\text{-L}^m\text{Ind} \vdash \neg\Phi\text{-L}^m\text{Ind}$
2. $\Phi\text{-PL}^m\text{Ind} \vdash \Phi\text{-L}^{m+1}\text{Ind}$
3. $\Phi\text{-L}^{m+1}\text{Ind} \vdash \Phi\text{-PL}^m\text{Ind}$
4. $\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-LInd} \vdash \mathrm{BB}\,\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}$
5. $\Sigma^{\mathrm{b}}_{n+1}\text{-PLInd} \vdash \mathrm{BB}\,\Sigma^{\mathrm{b}}_{n+1}$
6. $\mathrm{s}\Pi^{\mathrm{b}}_{n+1}\text{-PL}^m\text{Ind} \vdash \mathrm{s}\Sigma^{\mathrm{b}}_n\text{-L}^m\text{Ind}$
7. $\Pi^{\mathrm{b}}_{n+1}\text{-PL}^m\text{Ind} \vdash \Sigma^{\mathrm{b}}_n\text{-L}^m\text{Ind}$
8. $\Delta^{\mathrm{b}}_0\text{-PL}^{m+1}\text{Ind} \vdash \Delta^{\mathrm{b}}_0\text{-L}^{m+1}\text{Ind}$

Proof: All assertions except 4. are direct consequences of Lemma 8.2.1. To prove 4. let $F \in \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}$ and

$$G :\equiv (\exists w{\le}\, \mathrm{SqBd}(x_1, x_0)\,)(\forall y_0{\le}|x_0|\,)(y_0 \le z \to F_{y_1}(\beta(\mathrm{S}\, y_0, w))).$$

Applying sharply bounded collection yields

$$\mathrm{BB}\,\mathrm{s}\Pi^{\mathrm{b}}_n \vdash \text{"}(\forall y_0{\le}|x_0|\,)(y_0 \le z \to F_{y_1}(\beta(\mathrm{S}\, y_0, w))) \in \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{"},$$

hence

$$\mathrm{BB}\,\mathrm{s}\Pi^{\mathrm{b}}_n \vdash \text{"}G \in \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{"}.$$

Lemma 8.2.1 5. immediately shows

$$\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-LInd} \vdash \mathrm{BB}\,\mathrm{s}\Pi^{\mathrm{b}}_n$$

hence

$$\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-LInd} \vdash \text{"}G \in \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{"}.$$

Now Lemma 8.2.1 5. proves

$$\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-LInd} \vdash BB(F, y_0, y_1, x_0, x_1).$$

□

We use the last two lemmas to compare several theories:

8.2.3 Theorem *Let $m, n \ge 0$, then*

$$\begin{array}{ccc}
\mathrm{s}\Pi^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} & = & \mathrm{s}\Sigma^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} \\
\cap & & \\
\mathrm{s}\Pi^{\mathrm{b}}_{n+1}\text{-PL}^m\mathrm{Ind} & = & \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-PL}^m\mathrm{Ind} \\
\shortparallel & & \\
\mathrm{s}\Pi^{\mathrm{b}}_{n+1}\text{-L}^{m+1}\mathrm{Ind} & = & \mathrm{s}\Sigma^{\mathrm{b}}_{n+1}\text{-L}^{m+1}\mathrm{Ind}\,.
\end{array}$$

Furthermore

$$\mathrm{S}^0_2 = \Delta^{\mathrm{b}}_0\text{-L}^{m+1}\mathrm{Ind}$$

and

$$\begin{array}{l}
\mathrm{sR}_2^n = \mathrm{s}\Pi_n^{\mathrm{b}}\text{-LLInd} = \mathrm{s}\Pi_n^{\mathrm{b}}\text{-PLInd} = \mathrm{s}\Sigma_n^{\mathrm{b}}\text{-PLInd} \\
\bigcap \\
\mathrm{R}_2^n = \Pi_n^{\mathrm{b}}\text{-LLInd} = \Pi_n^{\mathrm{b}}\text{-PLInd} = \Sigma_n^{\mathrm{b}}\text{-PLInd} \overset{n\geq 0}{=} \mathrm{sR}_2^n + \mathrm{BB}\,\mathrm{s}\Sigma_n^{\mathrm{b}} \\
\bigcap \\
\mathrm{S}_2^n = \Pi_n^{\mathrm{b}}\text{-LInd} = \Pi_n^{\mathrm{b}}\text{-PInd} = \Sigma_n^{\mathrm{b}}\text{-PInd} \overset{n\geq 0}{=} \mathrm{S}_2^n + \mathrm{BB}\,\mathrm{s}\Sigma_n^{\mathrm{b}} \\
\shortparallel \\
\mathrm{s}\Pi_n^{\mathrm{b}}\text{-LInd} = \mathrm{s}\Pi_n^{\mathrm{b}}\text{-PInd} = \mathrm{s}\Sigma_n^{\mathrm{b}}\text{-LInd} = \mathrm{s}\Sigma_n^{\mathrm{b}}\text{-PInd} \subset \mathrm{sR}_2^{n+1} \\
\bigcap \\
\mathrm{T}_2^n = \Pi_n^{\mathrm{b}}\text{-Ind} = \mathrm{s}\Sigma_n^{\mathrm{b}}\text{-Ind} = \mathrm{s}\Pi_n^{\mathrm{b}}\text{-Ind} \subset \mathrm{S}_2^{n+1} \\
\shortparallel \ (n>0) \\
\mathrm{T}_2^n + \mathrm{BB}\,\mathrm{s}\Sigma_n^{\mathrm{b}}.
\end{array}$$

These connections directly relativize to $\mathcal{L}_{BA}(\mathcal{X})$. □

8.3 Comparing theories of BPA

8.3.1 Lemma ${}^{\mathrm{p}}$BASIC *proves*

1. $Ind(\neg F_y(x \dot{-} y), y, x) \rightarrow Ind(F, y, x)$
2. $PInd(F_y(|y|), y, x) \rightarrow LInd(F, y, x)$
3. $LInd(F_y(\mathrm{MSP}(x, |x| \dot{-} y)), y, x) \rightarrow PInd(F, y, x)$
4. $PInd((\forall y{\leq}x)(\forall u{\leq}x)(u{\leq}z{+}1 \wedge y{+}u{\leq}x \wedge F \rightarrow F_y(y{+}u)), z, x)$
   $\rightarrow Ind(F, y, x)$

8.3.2 Lemma *Let* $m, n \geq 0$

1. ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-L}^m\mathrm{Ind} \vdash {}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-L}^m\mathrm{Ind}$ *and*
   ${}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-L}^m\mathrm{Ind} \vdash {}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-L}^m\mathrm{Ind}$
2. ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-PL}^m\mathrm{Ind} \vdash {}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-L}^{m+1}\mathrm{Ind}$ *and*
   ${}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-PL}^m\mathrm{Ind} \vdash {}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-L}^{m+1}\mathrm{Ind}$
3. ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-L}^{m+1}\mathrm{Ind} \vdash {}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}\text{-PL}^m\mathrm{Ind}$ *and*
   ${}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-L}^{m+1}\mathrm{Ind} \vdash {}^{\mathrm{p}}\Pi_n^{\mathrm{b}}\text{-PL}^m\mathrm{Ind}$

4. ${}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}$-PL^{m}Ind $\vdash {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-L^{m}Ind

5. ${}^{\mathrm{p}}\Delta^{\mathrm{b}}_0$-PL^{m+1}Ind $\vdash {}^{\mathrm{p}}\Delta^{\mathrm{b}}_0$-L^{m+1}Ind

We use this to compare several theories:

8.3.3 Theorem *Let $m, n \geq 0$, then*

$$\begin{array}{ccc}
{}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} & = & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} \\
\bigcap & & \\
{}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}\text{-PL}^m\mathrm{Ind} & = & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-PL}^m\mathrm{Ind} \\
\shortparallel & & \\
{}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}\text{-L}^{m+1}\mathrm{Ind} & = & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-L}^{m+1}\mathrm{Ind}\,.
\end{array}$$

Furthermore

$${}^{\mathrm{p}}\mathrm{S}^0_2 = {}^{\mathrm{p}}\Delta^{\mathrm{b}}_0\text{-L}^{m+1}\mathrm{Ind}$$

and

$$\begin{array}{lllllllll}
{}^{\mathrm{p}}\mathrm{R}^n_2 & = & {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-LLInd} & = & {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-PLInd} & = & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-PLInd} & & \\
\bigcap & & & & & & & & \\
{}^{\mathrm{p}}\mathrm{S}^n_2 & = & {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-LInd} & = & {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-PInd} & = & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-PInd} & \subset & {}^{\mathrm{p}}\mathrm{R}^{n+1}_2 \\
\bigcap & & & & & & & & \\
{}^{\mathrm{p}}\mathrm{T}^n_2 & = & {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n\text{-Ind} & & & & & \subset & {}^{\mathrm{p}}\mathrm{S}^{n+1}_2.
\end{array}$$

These connections directly relativize to $\mathcal{L}_{BPA}(\mathcal{X})$. □

8.4 Comparing BA with BPA

8.4.1 Lemma ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-L^{m}Ind *is a conservative extension of* sΣ^{b}_n-L^{m}Ind.

Proof: By definition $\mathcal{L}_{BA}$ is a sub-language of $\mathcal{L}_{BPA}$ and sΣ^{b}_n-L^{m}Ind a subset of ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-L^{m}Ind. Thus, ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-L^{m}Ind is an extension of sΣ^{b}_n-L^{m}Ind. To prove that this extension is conservative let $v : \mathcal{L}_{BPA} \to \mathcal{L}_{BA}$ be the following transformation: $x_i \mapsto x_{2\cdot i}$, $a_i \mapsto x_{2\cdot i+1}$, v is identical on the other symbols of $\mathcal{L}_{BPA}$ and homeomorphic on terms and formulas. An easy induction on the derivation shows

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} \vdash F \implies \mathrm{s}\Sigma^{\mathrm{b}}_n\text{-L}^m\mathrm{Ind} \vdash F^v.$$

For $\mathcal{L}_{BA}$-sentences F we obviously have $\vdash F^v \leftrightarrow F$. □

The arguments directly relativize to $\mathcal{L}_{BPA}(\mathcal{X})$.

8.4.2 Lemma *Let* $m, n \geq 0$, *then* ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-L}^m\mathrm{Ind}$ *is a conservative extension of* $\mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-L}^m\mathrm{Ind}$. □

8.4.3 Corollary *Let* $n \geq 0$, *then*

${}^{\mathrm{p}}\mathrm{R}^n_2$ *is a conservative extension of* sR^n_2
${}^{\mathrm{p}}\mathrm{S}^n_2$ *is a conservative extension of* S^n_2
${}^{\mathrm{p}}\mathrm{T}^n_2$ *is a conservative extension of* T^n_2
${}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X})$ *is a conservative extension of* $\mathrm{sR}^n_2(\mathcal{X})$
${}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X})$ *is a conservative extension of* $\mathrm{S}^n_2(\mathcal{X})$
${}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X})$ *is a conservative extension of* $\mathrm{T}^n_2(\mathcal{X})$. □

Chapter 9

Well-ordering Proofs in BPA

In this chapter we transfer the well-ordering proofs of $\mathrm{I}\Sigma^0_n$ from Chapter 3 to bounded predicative arithmetic theories.

9.1 Formalization of wellfoundedness

In Chapter 6 we defined exponential notations (denoted by α, β, γ etc), the predicates $\mathcal{E}$, $\prec$ and the functions $\Phi_{\mathcal{E}}$, $\mathrm{T}_{\mathcal{E}}$, $\hat{+}$, $\hat{2}$ operating with exponential notations. Furthermore, we observed that the predicates $\mathcal{E}$, $\prec$ and functions $\hat{+}$, $\hat{2}$, $\mathrm{T}_{\mathcal{E}}$ are polytime.

The language $\mathcal{L}_{BPA}(\mathcal{X})$ contains the predicate symbols $\mathcal{E}$, $\prec$, $\mathcal{G}_{\hat{+}}$, $\mathcal{G}_{\hat{2}^{\cdot}}$, $\mathcal{G}_{\mathrm{T}_{\mathcal{E}}}$. To use exponential notations in formulas we abbreviate

$$
\begin{aligned}
\forall\beta\, A(\beta) &\;:\equiv\; \forall\beta\,\big(\beta\in\mathcal{E}\to A(\beta)\big)\\
\exists\beta\, A(\beta) &\;:\equiv\; \exists\beta\,\big(\beta\in\mathcal{E}\wedge A(\beta)\big)\\
\forall\beta\prec\alpha\, A(\beta) &\;:\equiv\; \forall\beta\,\big(\beta\prec\alpha\to A(\beta)\big)\\
\exists\beta\prec\alpha\, A(\beta) &\;:\equiv\; \exists\beta\,\big(\beta\prec\alpha\wedge A(\beta)\big)\\
(\forall u\, A)^t &\;:\equiv\; \forall u{\le}t\, A^t\\
(\exists u\, A)^t &\;:\equiv\; \exists u{\le}t\, A^t.
\end{aligned}
$$

For $\alpha\in\mathcal{E}$ the formula $Fund(\alpha,X)\;:\equiv\;Fund({\prec}{\restriction}\alpha,X)$ as defined in Chapter 3 expresses the wellfoundedness of $\prec$ up to α. But $Fund(\alpha,X)$ has the disadvantage that is contains unbounded quantifiers. These quantifiers can be bound because there is some $a\in\omega$ such that $\forall\beta\prec\alpha\,(\beta\le a)$ as the field$({\prec}{\restriction}\alpha)$ is a finite set. Then

$$Fund(\alpha,X)\iff Fund(\alpha,X)^a.$$

Therefore, we define

$$\begin{aligned} \alpha \sqsubset X &:\equiv \forall \beta \prec \alpha \big(\beta \in X\big) \\ Prog(a,\alpha,X) &:\equiv \big(\forall \beta \prec \alpha(\beta \sqsubset X \to \beta \in X)\big)^a \\ Fund(a,\alpha,X) &:\equiv Prog(a,\alpha,X) \to \big(\alpha \sqsubset X\big)^a. \end{aligned}$$

Observe that ${}^{\mathrm{p}}$BASIC proves $Fund(a,\alpha,X) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_2(\mathcal{X})$ which means that there is some $G \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_2(\mathcal{X})$ such that ${}^{\mathrm{p}}$BASIC $\vdash Fund(a,\alpha,X) \leftrightarrow G$. $(\prec\restriction\alpha) \cap (a \times a)$ is well-founded, hence $\mathbb{N} \models Fund(a,\alpha,X)$.

In general $\mathbb{N} \models Fund(a,\alpha,X)$ only states the wellfoundedness of $(\prec\restriction\alpha) \cap (a \times a)$ and does not express that $\prec\restriction\alpha$ is well-founded. To obtain the latter we need some a with $(\prec\restriction\alpha) \subset (a \times a)$. Usually we cannot take a to be α, because there is some α with $(\prec\restriction\alpha) \not\subset (\alpha\times\alpha)$, e.g. $\mathrm{T}_{\mathcal{E}}(7)^{\mathbb{N}} = 260322$ and $\mathrm{T}_{\mathcal{E}}(8)^{\mathbb{N}} = 65198$ ($\hat{2}^{\mathrm{T}_{\mathcal{E}}(x)}$ is a polytime function, hence polynomially bounded, thus there are unbounded many such α!). We formulate $(\prec\restriction\alpha) \subset (a\times a)$ in a bounded formula of $\mathcal{L}_{BPA}$. Then, assuming that a and α satisfy that formula, $\mathbb{N} \models Fund(a,\alpha,X)$ expresses the wellfoundedness of $\prec\restriction\alpha$. Also we express that the graphs $\mathcal{G}_f$ for $f \in \mathcal{F}^i$ define a total function on $a^{\#f}$. We do this by

$$\begin{aligned} Big(a,b,\alpha) :\equiv\ & \hat{1} \le a \wedge \alpha \le a \wedge \bigwedge_{f\in\mathcal{F}^i} \forall \vec{c}{\le}a\, \exists d{\le}b\, \mathcal{G}_f(\vec{c},d) \\ & \wedge \forall\beta{\le}a\, \forall\gamma{\le}a\, \forall\delta{\le}b\, \big([\mathcal{G}_{\hat{+}}(\beta,\gamma,\delta) \vee \mathcal{G}_{\hat{2}}(\beta,\delta)] \wedge \delta \preceq \alpha \to \delta \le a\big). \end{aligned}$$

Observe that ${}^{\mathrm{p}}$BASIC proves $Big(a,b,\alpha) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_2$. Obviously we have for $\alpha \le \beta$ and $\alpha \preceq \beta$ that $Big(a,b,\beta) \to Big(a,b,\alpha)$. The next lemma shows that $Big(a,b,\alpha)$ has the intended meaning.

9.1.1 Lemma *In the standard model* $\mathbb{N}$ *we have*

1. $\forall\alpha\, \exists a\, \exists b\, Big(a,b,\alpha)$.

2. $Big(a,b,\alpha) \implies \prec\restriction\alpha \subset a \times a$.

Proof: Let α be given. Then field$(\prec\restriction\alpha)$ is finite, thus there is some $a \ge \max\{\hat{1},\alpha\}$ such that $\prec\restriction\alpha \subset a \times a$. Now $\mathcal{F}^i$ contains only finitely many functions which on finite domains take only finitely many values. Thus, there is some b so that $\forall f \in \mathcal{F}^i\, \forall\vec{c} \le a(f(\vec{c}) \le b)$. Obviously a and b satisfy $Big(a,b,\alpha)$. Thus, we have shown the first assertion.

To prove the second one we set

$$\begin{aligned}
\Gamma_\alpha(X) &:= X \cup \{\hat{0}\} \cup \{\delta \preceq \alpha \,:\, \exists \beta, \gamma \in X\, (\delta = \hat{2}^{\beta} \text{ or } \delta = \beta \hat{+} \gamma)\} \\
\mathrm{I}_\alpha^{<n} &:= \bigcup_{k<n} \mathrm{I}_\alpha^k \\
\mathrm{I}_\alpha^k &:= \Gamma_\alpha\big(\mathrm{I}_\alpha^{<k}\big) \\
\mathrm{I}_\alpha &:= \mathrm{I}_\alpha^{<\omega} := \bigcup_{k<\omega} \mathrm{I}_\alpha^k.
\end{aligned}$$

Then Γ_α is a monotone operator, hence inductive, and I_α is the smallest fixed point of Γ_α. I_α satisfies

$$\mathrm{I}_\alpha = \{\beta : \beta \preceq \alpha\}$$

because obviously $\mathrm{I}_\alpha^{<n} \subset \{\beta : \beta \preceq \alpha\}$ for all n by induction on n. On the other hand we can show

$$\forall \beta \preceq \alpha\, \exists n\, \beta \in \mathrm{I}_\alpha^n$$

by induction on $\prec\restriction(\alpha+1)$. Let $\beta \preceq \alpha$. For $\beta = \hat{0}$ we know $\hat{0} \in \Gamma_\alpha(\emptyset) = \mathrm{I}_\alpha^1$. Otherwise, there are some $\beta_0 \prec \ldots \prec \beta_k \prec \beta$ with $\beta = \check{2}^{\beta_k} \check{+} \ldots \check{+} \check{2}^{\beta_0}$. The induction hypothesis yields some $n_0, \ldots, n_k$ with $\beta_i \in \mathrm{I}_\alpha^{n_i}$. For $n := \max\{n_i \,:\, i \le k\}$ we obtain $\beta_i \in \mathrm{I}_\alpha^n$, hence $\hat{2}^{\beta_i} \in \mathrm{I}_\alpha^{n+1}$, hence $\beta \in \mathrm{I}_\alpha^{n+k+1}$. □

In Lemma 6.3.3 we have shown that $|\mathrm{T}_{\mathcal{E}}(x)| \le 8 \cdot |x|^2 < 8 \cdot |x \# x|$, hence $\mathrm{T}_{\mathcal{E}}(x) \le (x \# x)^8$. Therefore, we compute for $\alpha \preceq \mathrm{T}_{\mathcal{E}}(x)$

$$\alpha \le (\Phi_{\mathcal{E}}(\alpha) \# \Phi_{\mathcal{E}}(\alpha))^8 \le (x \# x)^8.$$

Let $s(x) :\equiv \mathrm{SqBd}(x, (|x| \# |x|)^8)$. As $\mathcal{F}^i$ is a finite set of polytime functions we can find some $t(x)$ for $s(x)$ such that

$$\forall f {\in} \mathcal{F}^i\, \forall \vec{y} {\le} s(x)\, (f(\vec{y}) \le t(x)).$$

Thus, we have shown

9.1.2 Lemma *There are terms s, t with $\mathsf{FV}(s,t) \subset \{x\}$ such that $\mathbb{N} \models \forall x\, Big(s, t, \mathrm{T}_{\mathcal{E}}(x))$.* □

If, in the following proofs, we assume $Big(a, b, \alpha)$ then we can use $f \in \mathcal{F}^i$ as a function on the arguments $c_1, \ldots, c_{\mathrm{ar}(f)} \le a$. But we have to pay attention if an informal argument involves induction (because in this case the argument depends on the complexity of some formulas). This will be the case only when we come to Theorem 9.3.2.

Sometimes it will be convenient to consider an extended language $\mathcal{L}_{BPA}(\mathcal{X}, \mathcal{F}^i)$ of $\mathcal{L}_{BPA}(\mathcal{X})$ in which the function symbols $\underline{\mathrm{f}}$ for $f \in \mathcal{F}^i$ have arbitrary arguments - predicative and impredicative ones. In order to obtain from $\mathcal{L}_{BPA}(\mathcal{X}, \mathcal{F}^i)$-formulas F an $\mathcal{L}_{BPA}(\mathcal{X})$-formula such that the universal closure of this formula is equivalent to the universal closure of F in the standard model we define the *transformation* $el_{\mathcal{F}^i}$. $F^{el_{\mathcal{F}^i}}$ is the result of applying

$$G[\underline{\mathrm{f}}(\vec{s})] \quad :- \quad \mathcal{G}_f(\vec{s}, d_{f(\vec{s})}) \to G[d_{f(\vec{s})}]$$

for $f \in \mathcal{F}^i$, $\vec{s} \in \mathcal{L}_{BPA}(\mathcal{X})$ which contain only variables which are not under the scope of a quantifier in G, $\underline{\mathrm{f}}(\vec{s}) \notin \mathcal{L}_{BPA}(\mathcal{X})$ and $d_{f(\vec{s})}$ a new impredicative variable for $G[\underline{\mathrm{f}}(\vec{s})]$

as often as possible to F. (Read ": −" as "is replaced by".) To make this transformation well-defined we assume the applications to be ordered, e.g., we apply the rule always to the leftmost position in the string.

Notice: If $t \in \mathcal{L}_{BPA}(\mathcal{X}, \mathcal{F}^i)$ contains no safe variables then $t \in \mathcal{L}_{BPA}(\mathcal{X})$.

To give an example we compute the result of $el_{\mathcal{F}^i}$ as used in the assertion of Lemma 9.2.1:

$$\begin{aligned}&[Big(a,b,\alpha) \to Prog(a, \alpha \hat{+} \hat{1}, Pre(a))]^{el_{\mathcal{F}^i}}\\ &\quad\equiv \quad \mathcal{G}_{\hat{+}}(\alpha, \hat{1}, c) \wedge Big(a,b,\alpha) \to Prog(a, c, Pre(a))\end{aligned}$$

9.2 What $^{\mathrm{p}}$BASIC can prove

As we have seen in Chapter 6 the predecessor function $\hat{\mathrm{P}}$ on $\mathcal{E}$ does not have polynomial growth rate. Thus, we do not have a predecessor for all exponential notations. The set of all exponential notations for which some predecessor less than a exists is defined by

$$Pre(a) :\equiv \{\beta : \exists\gamma{\le}a\, (\mathcal{G}_{\hat{+}}(\gamma, \hat{1}, \beta) \vee \beta = \hat{0})\}.$$

Observe that $Pre(a) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_1$. Assuming $Big(a,b,\alpha)$ we can show that this set is progressive. Thus, $Big(a,b,\alpha)$ and $Fund(a, \alpha \hat{+} \hat{1}, Pre(a))$

imply that for all exponential notations β with $0 \prec \beta \preceq \alpha$ some predecessor of β exists.

9.2.1 Theorem

$$^{\mathrm{p}}\mathrm{BASIC} \vdash [Big(a,b,\alpha) \to Prog(a, \alpha \hat{+} \hat{1}, Pre(a))]^{el_{\mathcal{F}^i}}.$$

Proof: We argue in ${}^{\mathrm{p}}$BASIC. Assuming $Big(a,b,\alpha)$ and

$$\beta \leq a, \beta \prec \alpha \hat{+} \hat{1}, \tag{9.1}$$

$$\big(\beta \sqsubset Pre(a)\big)^a, \tag{9.2}$$

$\beta \neq \hat{0}$, we have to conclude $\exists \gamma \leq a(\beta = \gamma \hat{+} \hat{1})$. First we observe $\beta \preceq \alpha$. Using the axioms we obtain some $\xi, \eta \in \mathcal{E}$ with $\xi, \eta < \beta$ and $\beta = \xi \check{+} \check{2}^\eta$. We keep in mind that $\xi \leq a$ and $\xi \prec \beta \preceq \alpha$.

If $\eta = \hat{0}$ then we are done. Otherwise, $\eta \neq \hat{0}$, $\eta < \beta \leq a$ and $\eta \prec \hat{2}^\eta \preceq \beta$, thus using (9.2) we obtain some $\zeta \leq a$ with $\eta = \zeta \hat{+} \hat{1}$. Then $\zeta \prec \eta$, hence $\hat{2}^\zeta \prec \hat{2}^\eta \preceq \beta \preceq \alpha$, thus $Big(a,b,\alpha)$ yields $\hat{2}^\zeta \leq a$. Applying (9.2) yields some $\nu \leq a$ with $\hat{2}^\zeta = \nu \hat{+} \hat{1}$. Now we have to compute $\gamma := (\xi \hat{+} \hat{2}^\zeta) \hat{+} \nu \leq a$ and $\gamma \hat{+} \hat{1} = \beta$.

As $\xi, \hat{2}^\zeta \leq a$ and $\xi, \hat{2}^\zeta \preceq \alpha$ we obtain $\xi \hat{+} \hat{2}^\zeta \leq a$ using $Big(a,b,\alpha)$. We compute

$$(\xi \hat{+} \hat{2}^\zeta) \hat{+} \hat{2}^\zeta = \xi \hat{+} (\hat{2}^\zeta \hat{+} \hat{2}^\zeta) = \xi \hat{+} \hat{2}^{(\zeta \hat{+} \hat{1})} = \xi \hat{+} \hat{2}^\eta = \beta$$

hence $\xi \hat{+} \hat{2}^\zeta \prec (\xi \hat{+} \hat{2}^\zeta) \hat{+} \hat{2}^\zeta = \beta \preceq \alpha$. This and $\nu \leq a$, $\nu \prec \nu \hat{+} \hat{1} = \hat{2}^\zeta \preceq \alpha$ together with $Big(a,b,\alpha)$ imply $\gamma = (\xi \hat{+} \hat{2}^\zeta) \hat{+} \nu \leq a$. We finally compute

$$\gamma \hat{+} \hat{1} = (\xi \hat{+} \hat{2}^\zeta) \hat{+} (\nu \hat{+} \hat{1}) = (\xi \hat{+} \hat{2}^\zeta) \hat{+} \hat{2}^\zeta = \beta.$$

□

We want to prove $Fund(a, \hat{2}^\alpha, X)$ from $Fund(a, \alpha, Y)$ where Y is the set of all exponential notations $\beta \preceq \alpha$ such that we can jump from $\gamma \sqsubset X$ to $\gamma \hat{+} \hat{2}^\beta \sqsubset X$. This Y is called the *jump of* X and is defined by

$$\begin{aligned} Jp(a,\alpha,X) \;:\equiv\; \big\{\beta : \beta \preceq \alpha \wedge \big(&\forall\gamma\,\forall\delta_0\,\forall\delta_1\,\forall\delta_2\,(\mathcal{G}_{\hat{2}}(\beta,\delta_0) \wedge \mathcal{G}_{\hat{+}}(\gamma,\delta_0,\delta_1) \\ &\wedge \mathcal{G}_{\hat{2}}(\alpha,\delta_2) \wedge \delta_1 \preceq \delta_2 \wedge \gamma \sqsubset X \to \delta_1 \sqsubset X)\big)^a\big\} \end{aligned}$$

Observe that ${}^{\mathrm{p}}$BASIC proves for $A(b) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}$

$$Jp(a,\alpha,A(.)) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+2}.$$

9.2.2 Lemma

$$^{\mathrm{p}}\mathrm{BASIC} \vdash [Big(a,b,\hat{2}^{\alpha}) \wedge Fund(a,\alpha,Pre(a)) \wedge Fund(a,\alpha,Jp(a,\alpha,X)) \rightarrow Fund(a,\hat{2}^{\alpha},X)]^{el_{\mathcal{F}i}}.$$

Proof: We argue in $^{\mathrm{p}}$BASIC.
Assuming $Big(a,b,\hat{2}^{\alpha})$, $Fund(a,\alpha,Pre(a))$, $Fund(a,\alpha,Jp(a,\alpha,X))$ and $Prog(a,\hat{2}^{\alpha},X)$ we obtain with the following Lemma

$$Prog(a,\alpha\hat{+}\hat{1},Jp(a,\alpha,X)), \tag{9.3}$$

thus also $Prog(a,\alpha,Jp(a,\alpha,X))$. With $Fund(a,\alpha,Jp(a,\alpha,X))$ we see $\big(\alpha \sqsubset Jp(a,\alpha,X)\big)^{a}$, so an application of (9.3) yields

$$\alpha \in Jp(a,\alpha,X) \tag{9.4}$$

as $\alpha \prec \alpha\hat{+}\hat{1}$. We observe $\big(\hat{0} \sqsubset X\big)^{a}$ and $\hat{0}\hat{+}\hat{2}^{\alpha} = \hat{2}^{\alpha}$, so (9.4) produces

$$\hat{2}^{\alpha} \sqsubset X$$

and we are done. □

9.2.3 Lemma

$$^{\mathrm{p}}\mathrm{BASIC} \vdash [Big(a,b,\hat{2}^{\alpha}) \wedge Fund(a,\alpha,Pre(a)) \wedge Prog(a,\hat{2}^{\alpha},X) \rightarrow Prog(a,\alpha\hat{+}\hat{1},Jp(a,\alpha,X))]^{el_{\mathcal{F}i}}.$$

Proof: We argue in $^{\mathrm{p}}$BASIC.
First we assume $Big(a,b,\hat{2}^{\alpha})$ and $Fund(a,\alpha,Pre(a))$. As $\alpha \prec \hat{2}^{\alpha}$ and $\alpha < \hat{2}^{\alpha}$ we obtain $Big(a,b,\alpha)$, hence $Prog(a,\alpha\hat{+}\hat{1},Pre(a))$ by Theorem 9.2.1. By $Fund(a,\alpha,Pre(a))$ we obtain $\big(\alpha \sqsubset Pre(a)\big)^{a}$, thus using $Prog(a,\alpha\hat{+}\hat{1},Pre(a))$ again yields

$$\big(\forall\beta \preceq \alpha(\hat{0} \prec \beta \rightarrow \exists\gamma(\beta = \gamma\hat{+}\hat{1}))\big)^{a}. \tag{9.5}$$

Now we assume

$$Prog(a,\hat{2}^{\alpha},X) \tag{9.6}$$

$$\beta \leq a, \beta \preceq \alpha \tag{9.7}$$

$$\big(\beta \sqsubset Jp(a,\alpha,X)\big)^{a} \tag{9.8}$$

and have to conclude $\beta \in Jp(a, \alpha, X)$. Therefore, we assume

$$\gamma \leq a, \gamma \hat{+} \hat{2}^{\beta} \preceq \hat{2}^{\alpha}, \left(\gamma \sqsubset X\right)^{a} \tag{9.9}$$

and now have to show that $\left(\gamma \hat{+} \hat{2}^{\beta} \sqsubset X\right)^{a}$. To do this we assume

$$\delta \leq a, \delta \prec \gamma \hat{+} \hat{2}^{\beta} \tag{9.10}$$

and derive $\delta \in X$.

We distinguish several cases:

$\beta = \hat{0}$: $\delta \prec \gamma$: With (9.10) $\delta \leq a$ we can use (9.9) to see $\delta \in X$.

$\delta \not\prec \gamma$: Using (9.10) we observe $\gamma \preceq \delta \prec \gamma \hat{+} \hat{1}$, hence $\delta = \gamma$ and we see from (9.9) $\delta \leq a, \delta \prec \delta \hat{+} \hat{1} \preceq \hat{2}^{\alpha}, \left(\delta \sqsubset X\right)^{a}$. Now we can apply (9.6) to derive $\delta \in X$.

$\beta \succ \hat{0}$: First we use (9.7) and (9.5) to obtain $\mu \leq a$ with $\beta = \mu \hat{+} \hat{1}$. So $\mu \prec \beta, \mu \leq a$ and (9.8) shows

$$\mu \in Jp(a, \alpha, X) \tag{9.11}$$

Rewriting (9.9) $\gamma \leq a, \left(\gamma \sqsubset X\right)^{a}, \gamma \hat{+} \hat{2}^{\mu} \prec \gamma \hat{+} \hat{2}^{\beta} \preceq \hat{2}^{\alpha}$ we can use $Big(a, b, \hat{2}^{\alpha})$ to obtain $\gamma \hat{+} \hat{2}^{\mu} \leq a$ and (9.11) to obtain

$$\left(\gamma \hat{+} \hat{2}^{\mu} \sqsubset X\right)^{a}. \tag{9.12}$$

Now we observe

$$(\gamma \hat{+} \hat{2}^{\mu}) \hat{+} \hat{2}^{\mu} = \gamma \hat{+} (\hat{2}^{\mu} \hat{+} \hat{2}^{\mu}) = \gamma \hat{+} \hat{2}^{\mu \hat{+} \hat{1}} = \gamma \hat{+} \hat{2}^{\beta} \preceq \hat{2}^{\alpha},$$

thus (9.12) and (9.11) imply

$$\left(\gamma \hat{+} \hat{2}^{\beta} = (\gamma \hat{+} \hat{2}^{\mu}) \hat{+} \hat{2}^{\mu} \sqsubset X\right)^{a},$$

hence $\delta \in X$ by (9.10).

□

It is surprising that in contrast to the well-ordering proof of $\mathrm{I}\Sigma^0_n$ the previous lemma is provable without any use of induction. The reason for this difference is that in the well-ordering proof of $\mathrm{I}\Sigma^0_n$ we use

$$\gamma + \omega^{\beta+1} \subset X \iff \forall k \in \omega(\gamma + \omega^{\beta} \cdot k \subset X)$$

for ordinals β, γ less than ε_0 and then show $\gamma + \omega^\beta \cdot k \subset X$ by induction on k. Here, in the view of exponential notations β, γ, we know

$$\gamma \hat{+} \hat{2}^{\beta \hat{+} \hat{1}} \subset X \iff \forall k \leq 2(\gamma \hat{+} \hat{2}^{\beta} \cdot \mathrm{T}_{\mathcal{E}}(k) \subset X)$$

thus we can prove $\gamma \hat{+} \hat{2}^{\beta \hat{+} \hat{1}} \subset X$ in two steps from $\gamma \subset X$.

Next we observe that in the previous lemmas we could have used arbitrary abstraction terms of $\mathcal{L}_{BPA}(\mathcal{X})$ instead of X.

9.2.4 Lemma *Let $A(a)$ be a formula, then*

$${}^{\mathrm{p}}\mathrm{BASIC} \vdash \Gamma \implies {}^{\mathrm{p}}\mathrm{BASIC} \vdash \Gamma_X(A(.)).$$

Proof: We use induction on the derivation. The only interesting case is an equality axiom $s \neq t, s \notin X, t \in X \subset \Gamma$. But we obtain ${}^{\mathrm{p}}\mathrm{BASIC} \vdash s \neq t, \neg A(s), A(t)$ by induction on the generation of A. □

We want to define the iterations of the jump operator by

$$\begin{aligned} Jp_0(a, \alpha, X) &:\equiv X \\ Jp_{k+1}(a, \alpha, X) &:\equiv Jp(a, \alpha, Jp_k(a, \hat{2}^{\alpha}, X)) \end{aligned}$$

but the second equation does not define an $\mathcal{L}_{BPA}(\mathcal{X})$-formula as $\hat{2}^{\alpha}$ is no $\mathcal{L}_{BPA}$-term. Therefore, we equivalently set

$$\begin{aligned} Jp_{k+1}(a, \alpha, X) :\equiv \big\{\beta : \beta \preceq \alpha \wedge \big(&\forall \gamma \, \forall \delta_0 \, \forall \delta_1 \, \forall \delta_2 \, (\mathcal{G}_{\hat{2}}(\beta, \delta_0) \\ &\wedge \mathcal{G}_{\hat{+}}(\gamma, \delta_0, \delta_1) \wedge \mathcal{G}_{\hat{2}}(\alpha, \delta_2) \wedge \delta_1 \preceq \delta_2 \\ &\wedge \gamma \sqsubset Jp_k(a, \delta_2, X) \to \delta_1 \sqsubset Jp_k(a, \delta_2, X))\big)^a\big\} \end{aligned}$$

Observe that ${}^{\mathrm{p}}\mathrm{BASIC}$ proves for $A(c) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{m+1}$

$$Jp_k(a, \alpha, A(.)) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{k+m+1}.$$

9.2.5 Lemma

$$\begin{aligned} {}^{\mathrm{p}}\mathrm{BASIC} \quad \vdash \quad & [Big(a, b, \hat{2}^{\alpha}) \wedge \bigwedge_{k=0}^{j+1} Fund(a, \alpha, Jp_k(a, \alpha, Pre(a))) \\ & \to \bigwedge_{k=0}^{j} Fund(a, \hat{2}^{\alpha}, Jp_k(a, \hat{2}^{\alpha}, Pre(a)))]^{el_{\mathcal{F}^i}}. \end{aligned}$$

Proof: We argue in $^{\mathrm{p}}$BASIC and assume

$$Big(a, b, \hat{2}^{\alpha}) \tag{9.13}$$

$$Fund(a, \alpha, Pre(a)) \tag{9.14}$$

$$\textstyle\bigwedge_{k=0}^{j} Fund(a, \alpha, Jp(a, \alpha, Jp_k(a, \hat{2}^{\alpha}, Pre(a)))) \tag{9.15}$$

Using Lemma 9.2.2 and 9.2.4 with the assumptions (9.13) and (9.14) we conclude from (9.15)

$$\bigwedge_{k=0}^{j} Fund(a, \hat{2}^{\alpha}, Jp_k(a, \hat{2}^{\alpha}, Pre(a))).$$

□

9.2.6 Theorem $^{\mathrm{p}}$BASIC *proves*

$$\begin{aligned} & Fund(a, \mathrm{T}_{\mathcal{E}}(x), Jp_i(a, \mathrm{T}_{\mathcal{E}}(x), X)) \\ & \wedge \textstyle\bigwedge_{k=0}^{i-1} Fund(a, \mathrm{T}_{\mathcal{E}}(x), Jp_k(a, \mathrm{T}_{\mathcal{E}}(x), Pre(a))) \\ & \wedge Big(a, b, \hat{2}_i(\mathrm{T}_{\mathcal{E}}(x))) \quad \rightarrow \quad Fund(a, \hat{2}_i(\mathrm{T}_{\mathcal{E}}(x)), X). \end{aligned}$$

Proof: We argue in $^{\mathrm{p}}$BASIC and assume

$$Fund(a, \mathrm{T}_{\mathcal{E}}(x), Jp_i(a, \mathrm{T}_{\mathcal{E}}(x), X)) \tag{9.16}$$

$$\textstyle\bigwedge_{k=0}^{i-1} Fund(a, \mathrm{T}_{\mathcal{E}}(x), Jp_k(a, \mathrm{T}_{\mathcal{E}}(x), Pre(a))) \tag{9.17}$$

and $Big(a, b, \hat{2}_i(\mathrm{T}_{\mathcal{E}}(x)))$. The last assumption yields

$$\forall j{\leq}i\, Big(a, b, \hat{2}_j(\mathrm{T}_{\mathcal{E}}(x))).$$

Thus, we obtain from (9.17) by successively applying Lemma 9.2.5

$$Fund(a, \hat{2}_k(\mathrm{T}_{\mathcal{E}}(x)), Pre(a))$$

for $k = 0, \ldots, i-1$. Using this, Lemma 9.2.2 and 9.2.4 yield

$$\begin{aligned} & Fund(a, \hat{2}_k(\mathrm{T}_{\mathcal{E}}(x)), Jp_{i-k}(a, \hat{2}_k(\mathrm{T}_{\mathcal{E}}(x)), X)) \rightarrow \\ & \qquad Fund(a, \hat{2}_{k+1}(\mathrm{T}_{\mathcal{E}}(x)), Jp_{i-(k+1)}(a, \hat{2}_{k+1}(\mathrm{T}_{\mathcal{E}}(x)), X)) \end{aligned}$$

for $k = 0, \ldots, i-1$, hence by (9.16)

$$Fund(a, \hat{2}_i(\mathrm{T}_{\mathcal{E}}(x)), X).$$

□

9.3 Proving foundation by induction

9.3.1 Lemma *Let p be a monotone polynomial and $B \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X}) \cup {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}(\mathcal{X})$, then*

$$ {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-L}^m\text{Ind} \vdash \mathit{Ind}(B, y, p(|x|_m)). $$

Proof: The monotone polynomials in $|x|_m$, $\mathrm{MP}(|x|_m)$, can be defined inductively by

1. $0 \in \mathrm{MP}(|x|_m)$
2. IF $p \in \mathrm{MP}(|x|_m)$ then $(p + 1) \in \mathrm{MP}(|x|_m)$.
3. If $p \in \mathrm{MP}(|x|_m)$ then $p \cdot |x|_m \in \mathrm{MP}(|x|_m)$.

We prove by induction on this generation

$$ \forall p \in \mathrm{MP}(|x|_m) \quad \forall B \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X}) \quad {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-L}^m\text{Ind} \vdash \mathit{Ind}(B, y, p), $$

then Lemma 8.3.1 1. yields the assertion. In case 1. the assertion directly follows. Arguing in ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-L^{m}Ind for the second case we assume $B \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$, $B_y(0)$ and

$$ \forall y{<}(p + 1)\, (B \to B_y(y + 1)). \tag{9.18} $$

By the induction hypothesis $B(p)$, hence $B(p + 1)$ by 9.18.

In the third case let $B \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$. Arguing in ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-L^{m}Ind we assume $B_y(0)$ and

$$ \forall y{<}p \cdot |x|_m\, (B \to B_y(y + 1)), \tag{9.19} $$

then we have to show that $B_y(p \cdot |x|_m)$. Let

$$ C \;:\equiv\; B_y(y \cdot |x|_m) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X}). $$

The induction hypothesis yields $\mathit{Ind}(C, y, p)$. Now $C_y(0) \leftrightarrow B_y(0)$ and $C_y(p) \;\equiv\; B_y(p \cdot |x|_m)$, thus it suffices to show that

$$ \forall y{<}p\, (C \to C_y(y + 1)). \tag{9.20} $$

Let $D \;:\equiv\; B_y(y \cdot |x|_m + z) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$. In order to show (9.20) let $y < p$ and assume C, that is $D_z(0)$. From (9.19) we obtain

$$ \forall z{<}|x|_m\, (D \to D_z(z + 1)), $$

thus ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-$\mathrm{L}^m$Ind proves $D_z(|x|_m)$, but this is

$$B_y(y \cdot |x|_m + |x|_m) \leftrightarrow B_y((y+1) \cdot |x|_m) \;\equiv\; C_y(y+1).$$

□

9.3.2 Theorem ${}^{\mathrm{p}}$BASIC *proves*

$$Ind\big([y \le x \to (\mathrm{T}_{\mathcal{E}}(y) \sqsubset X)^a], y, x\big) \to Fund(a, \mathrm{T}_{\mathcal{E}}(x), X).$$

Proof: Let $B(y) \;:\equiv\; y \le x \to (\mathrm{T}_{\mathcal{E}}(y) \sqsubset X)^a$. We argue in ${}^{\mathrm{p}}$BASIC. Assuming $Ind(B(y), y, x)$ and $Prog(a, \mathrm{T}_{\mathcal{E}}(x), X)$ we have to show that $\big(\mathrm{T}_{\mathcal{E}}(x) \sqsubset X\big)^a$, thus it suffices to show that $B(x)$ holds. We do this by induction on y in $B(y)$. Because $\mathrm{T}_{\mathcal{E}}(0) = \hat{0}$ and $\neg\alpha \prec \hat{0}$ holds for any α we obtain $B(0)$.

Now assume $B(y)$. We want to conclude $B(y+1)$, thus assuming $y{+}1 \le x, \alpha \le a, \alpha \prec \mathrm{T}_{\mathcal{E}}(y{+}1) = \mathrm{T}_{\mathcal{E}}(y) \mathbin{\hat{+}} \hat{1}$ we have to show that $\alpha{\in}X$. If $\alpha \prec \mathrm{T}_{\mathcal{E}}(y)$ this is obtained by $B(y)$. Otherwise, $\alpha = \mathrm{T}_{\mathcal{E}}(y) \prec \mathrm{T}_{\mathcal{E}}(x)$. From $B(y)$ we know $\big(\mathrm{T}_{\mathcal{E}}(y) \sqsubset X\big)^a$, hence $Prog(a, \mathrm{T}_{\mathcal{E}}(x), X)$ yields $\alpha \in X$, hence $B(y+1)$.

Now $Ind(B(y), y, x)$ yields $B(x)$. □

Observe that ${}^{\mathrm{p}}$BASIC proves for $A(b) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{l+1}(\mathcal{X})$

$$[y \le x \to \big(\mathrm{T}_{\mathcal{E}}(y) \sqsubset A(.)\big)^a] \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{l+1}(\mathcal{X}).$$

We abbreviate

$$BigFun(a, b, \alpha, X) \;:\equiv\; Big(a, b, \alpha) \to Fund(a, \alpha, X)$$

and observe that ${}^{\mathrm{p}}$BASIC proves $BigFun(a, b, \alpha, X) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_2(\mathcal{X})$.

9.3.3 Theorem *Let p be a monotone polynomial and $m, n \ge 0$, then*

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun(a, b, \hat{2}_n(\mathrm{T}_{\mathcal{E}}(p(|x|_m))), X).$$

Proof: We have $Jp_n(a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), X) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}(\mathcal{X})$ as remarked before and

$$Jp_k(a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), Pre(a)) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}$$

for $k = 0, \dots, n-1$ as $Pre(a) \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_1$. By Theorem 9.3.1 we obtain ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind} \vdash Ind(B(y), y, p(|x|_m))$ for all $B(y) \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_{n+1}(\mathcal{X})$, hence by Theorem 9.3.2

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} \vdash Fund[a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), Jp_k(a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), Pre(a))]$$

for all $k < n$, and

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} \vdash Fund[a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), Jp_n(a, \mathrm{T}_{\mathcal{E}}(p(|x|_m)), X)].$$

Now Theorem 9.2.6 yields the assertion. □

Let $X(d) :\equiv \{\varphi : \mathrm{Bit}(\varphi, d)\}$. By Theorem 7.2.6 and the preceding Theorem we obtain

9.3.4 Theorem *Let p be a monotone polynomial and $m, n \geq 0$, then*

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun[a, b, \hat{2}_n(\mathrm{T}_{\mathcal{E}}(p(|x|_m))), X(d)].$$

Chapter 10

A Semi-formal System for BPA

In the first part of this thesis from Chapter 3 to Chapter 5 we have investigated the truth complexity $\mathrm{tc}(F)$ of Π^1_1-sentences F of $\mathcal{L}_{Z_1}$. With the Boundedness Theorem 5.2.5 we observed that in some special cases $\mathrm{tc}(F)$ is connected with the meaning of F: if F states the wellfoundedness of some relation $\prec$ then the order-type of $\prec$ is bounded by $\mathrm{tc}(F)$. Thus, the estimation of the truth complexity yielded a characterization of the sentences provable in the fragments $\mathrm{I}\Sigma^0_n$ of Z_1.

In the present Chapter we will develop a suitable machinery to examine similarly fragments of BPA. As the definition of the truth complexity of Π^1_1-sentences bases on semi-formal systems we first define a semi-formal system for BPA and try to find a notion like the truth complexity for BPA which enables us to characterize the sentences provable in fragments of BPA.

10.1 $\mathcal{L}^p_\omega$ and the semi-formal system $\mathrm{BSF}^{\mathrm{p}}$

We begin with a definition of a *predicative version of the infinitary language* $\mathcal{L}_\infty$. The basic symbols of $\mathcal{L}^p_\omega$ consists of the logical symbols: $a_0, a_1, \ldots,\ X_0, X_1, \ldots,\ \bigwedge, \bigvee, \forall, \exists, =, \neq, \in, \notin$, and the same non-logical symbols as $\mathcal{L}_{BPA}$. The terms of $\mathcal{L}^p_\omega$ are the predicative ground terms of $\mathcal{L}_{BPA}$ (i.e., the ground terms of $\mathcal{L}_{BPA}$ and $a_0, a_1, \ldots$). The atomic formulas of $\mathcal{L}^p_\omega$ are the predicative ground atomic formulas of $\mathcal{L}_{BPA}$. With these all $\mathcal{L}^p_\omega$-formulas are generated by:

If $0 < n < \omega$ and $(A_i)_{i \leq n}$ is a sequence of $\mathcal{L}^p_\omega$-formulas then $\bigwedge_{i \leq n} A_i$ and $\bigvee_{i \leq n} A_i$ also are $\mathcal{L}^p_\omega$-formulas. And if $A(a)$ is an $\mathcal{L}^p_\omega$-formula and s is a predicative ground $\mathcal{L}_{BPA}$-term then $\forall a{\leq}s\, A(a)$ and $\exists a{\leq}s\, A(a)$ are $\mathcal{L}^p_\omega$-formulas.

Similar to Chapter 4 negation is not a logical symbol but can be defined as a syntactic operation. We define the *canonical translation* $*_p$ of the predicative ground formulas in $\mathrm{PBF}(\mathcal{X})$ to $\mathcal{L}^p_\omega$ by:

1. $F^{*_p} :\equiv F$ if F is an atomic formula
2. $(F_0 \wedge F_1)^{*_p} :\equiv \bigwedge_{i \leq 1} F_i^{*_p}$,
3. $(F_0 \vee F_1)^{*_p} :\equiv \bigvee_{i \leq 1} F_i^{*_p}$,
4. $(\forall x{\leq}t\, F(x))^{*_p} :\equiv \bigwedge_{n \leq t^{\mathbb{N}}} F(\underline{n})^{*_p}$,
5. $(\exists x{\leq}t\, F(x))^{*_p} :\equiv \bigvee_{n \leq t^{\mathbb{N}}} F(\underline{n})^{*_p}$,
6. $(\forall a{\leq}s\, F(a))^{*_p} :\equiv \forall a{\leq}s\, F(a)^{*_p}$,
7. $(\exists a{\leq}s\, F(a))^{*_p} :\equiv \exists a{\leq}s\, F(a)^{*_p}$.

For case 4. and 5. in this definition remember that x assigns a normal variable, thus t has to be a normal term by the definition of $\mathrm{PBF}(\mathcal{X})$. Therefore, t is predicative ground by the assumption that the translated formula is predicative ground. But this means that t is ground, thus we can compute $t^{\mathbb{N}}$.

For $\Gamma \subset \mathrm{PBF}(\mathcal{X})$ we define $\Gamma^{*_p} := \{F^{*_p} : F \in \Gamma\}$.

We define the *predicative rank* $\mathrm{prk}(F)$ and the *predicative length* $\mathrm{plh}(F)$ of a $\mathcal{L}^p_\omega$-formula F.

10.1.1 Definition *For $F \in \mathcal{L}^p_\omega$ we define*

$$\mathrm{prk}(F) := \min\{n : F \in {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})^{*_p} \cup {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n(\mathcal{X})^{*_p}\} \cup \{\omega\}.$$

We will often compute upper bounds. These bounds should be monotone $\mathcal{L}_{BPA}$-terms. To obtain this we can define a meta-function σ which assigns to each $\mathcal{L}_{BPA}$-term t a monotone $\mathcal{L}_{BPA}$-term $\sigma[t]$ in the variables $\vec{x}$ of t satisfying $\forall \vec{x}\,(t \leq \sigma[t])$[1]. Substituting a monotone term into another monotone term yields again a monotone term. Therefore, it suffices to associate some monotone term to each polytime function.

[1] Cf. [6], p. 77.

This can easily be done because every polytime function is polynomially bounded and to each monotone polynomial $p(\vec{x})$ we can find a term t_p which is build up form $0, \#$ and the variables $\vec{x}$ of p and satisfies $p(|\vec{x}|) \leq |t_p|$.[2]

Let the *length of* $F \in {}^{\text{p}}\Delta_0^{\text{b}}(\mathcal{X})^{*_p}$ be defined as in Chapter 5. We will immediately define the *predicative length* $\text{plh}(F)$ for $F \in \text{PBF}(\mathcal{X})$ which will be a monotone $\mathcal{L}_{BPA}$-term in the normal variables of F. The binary length of this $|\,\text{plh}(F)|$ will bound the length of all ${}^{\text{p}}\Delta_0^{\text{b}}(\mathcal{X})$-sub-formulas of F (i.e., the number of atomic formulas in the $*_p$ translation of every ${}^{\text{p}}\Delta_0^{\text{b}}(\mathcal{X})$-sub-formula of F).

10.1.2 Definition *Let* $F \in \text{PBF}(\mathcal{X})$*. We inductively define* $\text{plh}(F)$ *by the following clauses:*

1. *If F is atomic, let* $\text{plh}(F) :\equiv 1$.
2. *If* $F \equiv G \circ H$, $\circ \in \{\wedge, \vee\}$*, let* $\text{plh}(F) :\equiv 2 \cdot \text{plh}(G) \cdot \text{plh}(H)$.
3. *Assume* $F \equiv Qx{\leq}t\, A(x)$ *with* $Q \in \{\forall, \exists\}$ *and x normal. Let* $s :\equiv \text{plh}(A)_x(\sigma[t])$*. If* $F \notin {}^{\text{p}}\Delta_0^{\text{b}}(\mathcal{X})$*, let* $\text{plh}(F) :\equiv s$*. Otherwise, there is some $\mathcal{L}_{BPA}$-term t' with* $t \equiv |t'|$*. Let* $\text{plh}(F) :\equiv s \# \sigma[\text{S}_1(t')]$.
4. *If* $F \equiv Qa{\leq}s\, A(a)$, $Q \in \{\forall, \exists\}$ *and a impredicative, let* $\text{plh}(F) :\equiv \text{plh}(A(a))$.

Observe that $\text{plh}(F)$ is a monotone $\mathcal{L}_{BPA}$-term in the normal variables of the formula $F \in \text{PBF}(\mathcal{X})$. Therefore, if F is predicative ground then $\text{plh}(F)$ is a ground term.

The following lemma shows that $\text{plh}(\cdot)$ has the intended meaning.

10.1.3 Lemma *Let* $F \in \text{PBF}(\mathcal{X})$ *be predicative ground.*

1. *If* $F \notin {}^{\text{p}}\Delta_0^{\text{b}}(\mathcal{X})$ *then*

$$
\begin{aligned}
F \equiv G_0 \circ G_1 &\implies |\,\text{plh}(F)| \geq |\,\text{plh}(G_i)| \quad for \quad i \leq 1 \\
F \equiv Qx{\leq}t\, A(x) &\implies |\,\text{plh}(F)| \geq |\,\text{plh}\, A(\underline{n})| \quad for \quad n \leq t^{\mathbb{N}} \\
F \equiv Qa{\leq}s\, A(a) &\implies |\,\text{plh}(F)| \geq |\,\text{plh}(A(t))| \\
&\qquad\qquad for\ any\ term\ t
\end{aligned}
$$

[2]Cf. Chapter 2.

2. *If* $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})$ *then*

$$\begin{aligned} F \quad \textit{atomic} &\implies |\operatorname{plh}(F)| = 1 \\ F \equiv G_0 \circ G_1 &\implies |\operatorname{plh}(F)| \geq |\operatorname{plh}(G_0)| + |\operatorname{plh}(G_1)| \\ F \equiv Qx{\leq}|t|\, A(x) &\implies |\operatorname{plh}(F)| \geq \sum_{n \leq |t|^{\mathbb{N}}} |\operatorname{plh} A(\underline{n})| \end{aligned}$$

Notice: $|\operatorname{plh}(F)|$ is an upper bound of the length $\operatorname{lh}(F^{*_p})$ for formulas $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})$.

Analogous to Chapter 4 we define a semi-formal system ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma$ which has the meaning that there is a finitary proof tree (build up by special rules defined below) with the depth bounded by m, the predicative rank of each cut-formula strictly bounded by r and the binary length of the predicative length of each cut-formula bounded by l.

10.1.4 Definition *Let* $m, r, l < \omega$ *and* Γ *be a finite set of* $\mathcal{L}^p_\omega$*-formulas. We inductively define the* predicative version of a bounded semi-formal system BSF$^{\mathrm{p}}$ *by the following clauses.*

(Ax1) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma$ *holds if* Γ *contains a ground atomic formula which is true.*

(Ax2) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma$ *holds if* Γ *contains a logical axiom* $\neg A, A$ *or an equality axiom of the form* $s = s$*, or of the form* $s_0 \neq s_1, \neg A(t_0), A(t_1)$ *if* $s_i \equiv t_i$ *or* $s_i^{\mathbb{N}} = t_i^{\mathbb{N}}$*, for terms* s, s_0, s_1, t_0, t_1 *and atomic formulas* A.

(AxM) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma$ *holds if* Γ *contains an instance of a formula from the set* ${}^{\mathrm{p}}$BASIC.

($\bigwedge$) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma, \bigwedge_{i\leq n} F_i$ *holds if there is some* $m' < m$ *with* ${}^{\mathrm{p}}\!\vdash^{m'}_{r,l} \Gamma, F_i$ *for all* $i \leq n$.

($\bigvee$) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma, \bigvee_{i\leq n} F_i$ *holds if there is some* $m' < m$ *and* $i_0 \leq n$ *with* ${}^{\mathrm{p}}\!\vdash^{m'}_{r,l} \Gamma, F_{i_0}$.

($\forall\leq$) ${}^{\mathrm{p}}\!\vdash^{m}_{r,l} \Gamma, \forall a{\leq}s\, F(a)$ *holds if there is some* $m' < m$ *and some impredicative variable* b *not occurring in* $\Gamma, \forall a{\leq}s\, F(a)$ *with* ${}^{\mathrm{p}}\!\vdash^{m'}_{r,l} \Gamma, b \not\leq s, F(b)$.

(∃≤) $^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, s \nleq t, \exists a{\leq}t\, F(a)$ *holds if there is some* $m' < m$ *with* $^{\mathrm{p}}\!\mid\!\frac{m'}{r,l}\, \Gamma, F(s)$.

(Cut) $^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma$ *holds if there is some* $m' < m$ *and some formula* F *with* $\mathrm{prk}(F) < r, |\,\mathrm{plh}(F)| \leq l$ *and* $^{\mathrm{p}}\!\mid\!\frac{m'}{r,l}\, \Gamma, F$ *and* $^{\mathrm{p}}\!\mid\!\frac{m'}{r,l}\, \Gamma, \neg F$.

The basic properties of this system are easily proved by induction on m:

Structural Rule Let $\Gamma \subset \Gamma', m \leq m', r \leq r', l \leq l'$, then
$^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma \implies {}^{\mathrm{p}}\!\mid\!\frac{m'}{r',l'}\, \Gamma'$.

Equality Lemma Let s, t be ground terms, $s^{\mathbb{N}} = t^{\mathbb{N}}$, then
$^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, F(s) \implies {}^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, F(t)$.

Substitution Rule Let a be safe, s any predicative ground term, then
$^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma \implies {}^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma_a(s)$.

($\bigwedge$)-Inversion $^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, \bigwedge_{i\leq n} F_i \implies {}^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, F_i$ for all $i \leq n$.

(∀)-Inversion $^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, \forall a{\leq}t\, F(a) \implies {}^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, s \nleq t, F(s)$ for all terms s.

($\bigvee$)-Exportation $^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, \bigvee_{i\leq n} F_i \implies {}^{\mathrm{p}}\!\mid\!\frac{m}{r,l}\, \Gamma, F_0, \ldots, F_n$.

The semi-formal system gives us the possibility to measure the truth complexity of formulas in $\mathrm{PBF}(\mathcal{X})$. Using the method of search trees[3] we obtain the following completeness result for predicative ground formulas $F \in \mathrm{PBF}(\mathcal{X})$:

$$\mathbb{N} \models F \iff \exists m{<}\omega\ {}^{\mathrm{p}}\!\mid\!\frac{m}{1,1}\, F^{*_p}.$$

We define the *predicative truth complexity* of a predicative ground formula $F \in \mathrm{PBF}(\mathcal{X})$ by

$$\mathrm{ptc}(F) := \begin{cases} \min\{m : {}^{\mathrm{p}}\!\mid\!\frac{m}{1,1}\, F^{*_p}\} & : \textit{if} \quad \mathbb{N} \models F \\ \omega & : \textit{otherwise}. \end{cases}$$

Of course it will be senseless for fragments $\mathcal{F} \subset \mathrm{PBF}(\mathcal{X})$ of predicative ground formulas to consider the usual "Π^1_1-ordinal" which is defined by $\sup\{\mathrm{ptc}(F) : F \in \mathcal{F}\}$, because if $\mathcal{F}$ is non-pathological we

[3]See, e.g., [17] for an explanation of this method.

always have $\sup\{\mathrm{ptc}(F) : F \in \mathcal{F}\} = \omega$. Therefore, we consider the *dynamic predicative truth complexity* $\mathrm{dptc}(F, x_0, \ldots, x_{k-1})\colon \omega^k \to \omega$ for true formulas $F \in \mathrm{PBF}$ containing no normal variable not in the list $x_0, \ldots, x_{k-1}$, which is defined by

$$\mathrm{dptc}(F, x_0, \ldots, x_{k-1}) := \lambda \vec{n}.\, \mathrm{ptc}(F_{\vec{x}}(\underline{\vec{n}})).$$

10.2 The embedding into $\mathrm{BSF}^{\mathrm{p}}$

In order to investigate the dynamic predicative truth complexity we need some auxiliary semi-formal system. We define $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma$ which in addition to the clauses of ${}^{\mathrm{p}}\vdash^{m}_{r,l} \Gamma$ consists of the following *multi-cut* of at most k ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})^{*_p} \cup {}^{\mathrm{p}}\Pi^{\mathrm{b}}_n(\mathcal{X})^{*_p}$-formulas:

(MC^k_n) $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, \neg F_0, F_j$ holds if $0 < j \le k$ and there are some $m' < m$ and some $F_1, \ldots, F_{j-1}$ such that $\mathrm{prk}(F_i) \le n$ and $|\,\mathrm{plh}(F_i)| \le l$ for $i \le j$ and $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m',k}_{r,l} \Gamma, \neg F_i, F_{i+1}$ for $i < j$.

The following basic properties of the auxiliary semi-formal system are again easily proved by induction on m:

Structural Rule Let $\Gamma \subset \Gamma', n \le n', m \le m', k \le k', r \le r', l \le l'$, then $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma \implies \mathrm{MC}^{\mathrm{p}}_{n'}\vdash^{m',k'}_{r',l'} \Gamma'$.

Equality Lemma Let s, t be ground terms, $s^{\mathbb{N}} = t^{\mathbb{N}}$, then $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, F(s) \implies \mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, F(t)$.

Substitution Rule Let a be safe, s any term, then $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma \implies \mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma_a(s)$.

($\bigwedge$)-Inversion $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, \bigwedge_{i \le n} F_i \implies \mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, F_i$ for all $i \le n$.

($\forall$)-Inversion $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, \forall a{\le}t\, F(a) \implies \mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, s \not\le t, F(s)$ for all terms s.

($\bigvee$)-Exportation $\mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, \bigvee_{i \le n} F_i \implies \mathrm{MC}^{\mathrm{p}}_{n}\vdash^{m,k}_{r,l} \Gamma, F_0, \ldots, F_n$.

We connect this auxiliary system with the actual semi-formal system $\mathrm{BSF}^{\mathrm{p}}$.

10.2.1 Theorem *$k > 0$ and* $\mathrm{MC}_n^{\mathrm{p}}\vdash^{m,k}_{n+1,l}\Gamma \implies {}^{\mathrm{p}}\vdash^{m\cdot|k|}_{n+1,l}\Gamma$.

Proof: We use induction on m. If the last inference is not (MC_n^k) we obtain the assertion directly (from the induction hypothesis if $m > 0$) by the same inference, because $\lambda m.m \cdot |k|$ is strictly monotone. Otherwise, there are some j, m' with $0 < j \leq k$ and $m' < m$, and some $F_0, \ldots, F_j$ with $\mathrm{prk}(F_i) \leq n$, $\mathrm{lh}(F_i) \leq l$ for $i \leq j$ and $\neg F_0, F_j \in \Gamma$ such that

$$ {}^{\mathrm{p}}\vdash^{m'\cdot|k|}_{n+1,l}\Gamma, \neg F_i, F_{i+1} \quad \text{for } i < j $$

using the induction hypothesis.

Now we proceed using the following strategy, which we picture for $j = 7$:

$$ \dfrac{\dfrac{\dfrac{\neg F_0, F_1 \quad \neg F_1, F_2}{\neg F_0, F_2} \quad \dfrac{\neg F_2, F_3 \quad \neg F_3, F_4}{\neg F_2, F_4}}{\neg F_0, F_4} \quad \dfrac{\dfrac{\neg F_4, F_5 \quad \neg F_5, F_6}{\neg F_4, F_6} \quad \neg F_6, F_7}{\neg F_4, F_7}}{\neg F_0, F_7} $$

Thus, we obtain ${}^{\mathrm{p}}\vdash^{m'\cdot|k|+|j|}_{n+1,l}\Gamma, \neg F_0, F_j$, hence ${}^{\mathrm{p}}\vdash^{m\cdot|k|}_{n+1,l}\Gamma$. □

If $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p\}$ and $\mathrm{x}_0, \ldots, \mathrm{x}_p \in \omega$ (abbreviated by $\vec{\mathrm{x}} \in \omega$), then we define $t\langle\vec{\mathrm{x}}\rangle :\equiv t_{x_0,\ldots,x_p}(\underline{\mathrm{x}_0}, \ldots, \underline{\mathrm{x}_p})$. Analogously we define $F\langle\vec{\mathrm{x}}\rangle$ for formulas F and we set $\Gamma\langle\vec{\mathrm{x}}\rangle$ for sets of formulas Γ in the obvious way. We write shortly $F, G\langle\vec{\mathrm{x}}\rangle$ instead of $\{F, G\}\langle\vec{\mathrm{x}}\rangle$ if this does not confuse.

In the following we will often identify a ground term t with its evaluation $t^{\mathbb{N}}$. It will be clear from the context what is meant.

10.2.2 Theorem (Embedding) *Let* $\Gamma \subset \mathrm{PBF}(\mathcal{X})$ *be a finite set with* $\mathsf{nFV}(\Gamma) \subset \{x_0, \ldots, x_p\}$. *Let* $\mathcal{T}$ *be* ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}$-$\mathrm{L}^m\mathrm{Ind}$ *or* ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$. *Assume* $\mathcal{T} \vdash \Gamma$, *then*
$\exists d, r{<}\omega \quad \exists \mathcal{L}_{BPA}$*-term* t *with* $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p\}$

$$ \forall \vec{\mathrm{x}} \in \omega \quad \mathrm{MC}_n^{\mathrm{p}}\vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle}\Gamma\langle\vec{\mathrm{x}}\rangle^{*_p}. $$

Proof: We consider only the case $\mathcal{T} = {}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$ because ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}$-$\mathrm{L}^m\mathrm{Ind} \subset {}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$. As remarked at the end of Chapter 7 we obtain a normal derivation $\mathcal{T} \vdash \Gamma$ in which all cut-formulas are

${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}(\mathcal{X})$-formulas and all formulas in the derivation are in $\mathrm{PBF}(\mathcal{X})$. We prove the assertion by induction along this derivation.

In the following we omit the superscript $*_p$. Observe that for every formula $F \in \mathrm{PBF}(\mathcal{X})$ containing no normal variable not in $\{x_0, \dots, x_p\}$ we can find some $d < \omega$ with

$$\forall \vec{\mathrm{x}} \in \omega \quad {}^{\mathrm{p}}\vdash^{d}_{0,0} \neg F, F\langle\vec{\mathrm{x}}\rangle \tag{10.1}$$

We distinguish the following cases concerning the last inference:

(AxL),(AxE),(Ax^{p}B) If Γ is a logical axiom, an equality axiom or an instance of an axiom from ${}^{\mathrm{p}}$BASIC then (Ax2), resp. (AxM) (and at most four ($\bigvee$)-inferences) yield ${}^{\mathrm{p}}\vdash^{4}_{0,0} \Gamma\langle\vec{\mathrm{x}}\rangle$ for any $\vec{\mathrm{x}} \in \omega$.

($\mathcal{T}$-IND) There is some ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n}(\mathcal{X})$-formula F and some normal term t' with $Ind(F, x, |t'|_m) \in \Gamma$. Let $t :\equiv |t'|_m$. If $x \notin \mathsf{FV}(F)$ then $F_x(0) \equiv F \equiv F_x(t)$, thus using (10.1) we obtain some $d < \omega$ with

$$\forall \vec{\mathrm{x}} \in \omega \quad {}^{\mathrm{p}}\vdash^{d}_{0,0} \neg F_x(0), F_x(t)\langle\vec{\mathrm{x}}\rangle$$

and three times ($\bigvee$) yields the assertion.

Otherwise, let $s' :\equiv \sigma[\mathrm{plh}(F)]_x(t) + t'$ and $s :\equiv |s'|_m$, then

$$\mathsf{FV}(s') = \mathsf{FV}(s) \subset \{x_0, \dots, x_p\}.$$

Using (10.1) and the Equality Lemma we obtain some d such that for any $\vec{\mathrm{x}} \in \omega$ and $i < t\langle\vec{\mathrm{x}}\rangle^{\mathbb{N}}$

$${}^{\mathrm{p}}\vdash^{d}_{0,0} \neg F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle, F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle \;\;\&\;\; {}^{\mathrm{p}}\vdash^{d}_{0,0} \neg F_x(\mathrm{S}\,\underline{i})\langle\vec{\mathrm{x}}\rangle, F_x(\underline{i+1})\langle\vec{\mathrm{x}}\rangle$$

hence

$${}^{\mathrm{p}}\vdash^{d+2}_{0,0} \exists x{<}t\,(F \wedge \neg F_x(\mathrm{S}\,x))\langle\vec{\mathrm{x}}\rangle, \neg F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle, F_x(\underline{i+1})\langle\vec{\mathrm{x}}\rangle$$

by ($\bigwedge$) and ($\bigvee$). Observe $t'\langle\vec{\mathrm{x}}\rangle \le s'\langle\vec{\mathrm{x}}\rangle$, hence $t\langle\vec{\mathrm{x}}\rangle \le s\langle\vec{\mathrm{x}}\rangle$, and for $i \le t\langle\vec{\mathrm{x}}\rangle$ we have $\mathrm{lh}(F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle) \le |\,\mathrm{plh}(F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle)| \le |s'|\langle\vec{\mathrm{x}}\rangle$ and $\mathrm{prk}(F_x(\underline{i})\langle\vec{\mathrm{x}}\rangle) \le n$. Therefore, we can apply ($\mathrm{MC}^{s\langle\vec{\mathrm{x}}\rangle}_{n}$) to produce (of course using Equality)

$${}^{\mathrm{p}}_{\mathrm{MC}_n}\vdash^{d+3,s\langle\vec{\mathrm{x}}\rangle}_{0,|s'|\langle\vec{\mathrm{x}}\rangle} \neg(\forall x{<}t\,(F \to F_x(\mathrm{S}\,x)))\langle\vec{\mathrm{x}}\rangle, \neg F_x(0)\langle\vec{\mathrm{x}}\rangle, F_x(t)\langle\vec{\mathrm{x}}\rangle.$$

Four times ($\bigvee$) yields

$$\forall \vec{\mathrm{x}} \in \omega \quad {}^{\mathrm{p}}_{\mathrm{MC}_n}\vdash^{d+7,s\langle\vec{\mathrm{x}}\rangle}_{0,|s'|\langle\vec{\mathrm{x}}\rangle} Ind(F, x, t)\langle\vec{\mathrm{x}}\rangle.$$

($\vee$) The assertion follows directly from the induction hypothesis.

($\wedge$) The assertion follows from the induction hypothesis after replacing the upper bounds by some common bounds (using Structural Rule). We may always take the sum of the inductively given normal terms. In the other cases we, therefore, will assume common upper bounds.

(Cut) There is some ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{\infty}(\mathcal{X})$-formula F such that $\mathsf{nFV}(F) \subset \{\vec{x}\}$, $\mathcal{T} \vdash \Gamma, F$ and $\mathcal{T} \vdash \Gamma, \neg F$. Thus, the induction hypothesis yields some $d, r < \omega$, and some $\mathcal{L}_{BPA}$-term t with

$$\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, F\langle\vec{\mathrm{x}}\rangle \quad \text{and} \quad \mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, \neg F\langle\vec{\mathrm{x}}\rangle$$

for all $\vec{\mathrm{x}} \in \omega$. Without loss of generality we may assume $r > \mathrm{prk}(F)$ and $t\langle\vec{\mathrm{x}}\rangle \geq \mathrm{plh}(F\langle\vec{\mathrm{x}}\rangle)$. Applying (Cut) produces the assertion.

($\exists$) There are some term s, some variable φ and some formula F with [φ normal $\implies$ s normal], $(\exists\varphi\, F) \in \Gamma$ and $\mathcal{T} \vdash \Gamma, F_{\varphi}(s)$. By assumption $(\exists\varphi\, F) \in \mathrm{PBF}(\mathcal{X})$, thus there has to be some $B \in \mathrm{PBF}(\mathcal{X})$ and some term u such that [φ normal $\implies$ u normal], $\exists\varphi\, F \equiv \exists\varphi{\leq}u\, B$ and $F_{\varphi}(s) \equiv s \leq u \wedge B_{\varphi}(s)$. The induction hypothesis and ($\bigwedge$)-Inversion produce some $d, r < \omega$, $r > 0$, some t with $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p\}$ and

$$\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, s \leq u\langle\vec{\mathrm{x}}\rangle \tag{10.2}$$

and

$$\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, B_{\varphi}(s)\langle\vec{\mathrm{x}}\rangle \tag{10.3}$$

for all $\vec{\mathrm{x}} \in \omega$.

Fix $\vec{\mathrm{x}} \in \omega$. If φ is safe, we apply ($\exists\leq$) to (10.3) and obtain

$$\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d+1,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, s \not\leq u, \exists\varphi{\leq}u\, B\langle\vec{\mathrm{x}}\rangle.$$

Now $\exists\varphi{\leq}u\, B \equiv \exists\varphi\, F \in \Gamma$, hence

$$\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d+2,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma\langle\vec{\mathrm{x}}\rangle$$

by a cut with (10.2).

If φ is normal, then s and u have to be normal, thus $s\langle\vec{\mathrm{x}}\rangle$ and $u\langle\vec{\mathrm{x}}\rangle$ are ground. If $s\langle\vec{\mathrm{x}}\rangle \not\leq u\langle\vec{\mathrm{x}}\rangle$ then (Ax1) yields $\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma, s \not\leq u\langle\vec{\mathrm{x}}\rangle$, hence by a (Cut) with (10.2) $\mathrm{MC}^{\mathrm{p}}_{n} \vdash^{d+1,|t|_m\langle\vec{\mathrm{x}}\rangle}_{r,|t|\langle\vec{\mathrm{x}}\rangle} \Gamma\langle\vec{\mathrm{x}}\rangle$.

Otherwise $s\langle\vec{x}\rangle \leq u\langle\vec{x}\rangle$. The Equality Lemma applied to (10.3) shows $\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d,|t|_m\langle\vec{x}\rangle}{r,|t|\langle\vec{x}\rangle}\right. \Gamma, B_\varphi(\underline{s\langle\vec{x}\rangle^{\mathbb{N}}})\langle\vec{x}\rangle$, hence by $(\bigvee)$

$$\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d+1,|t|_m\langle\vec{x}\rangle}{r,|t|\langle\vec{x}\rangle}\right. \Gamma,$$

because

$$\bigvee_{n\leq u\langle\vec{x}\rangle^{\mathbb{N}}} B_\varphi(\underline{n})\langle\vec{x}\rangle^{*_p} \;\equiv\; (\exists\varphi{\leq}u\, B\langle\vec{x}\rangle)^{*_p} \;\equiv\; (\exists\varphi\, F\langle\vec{x}\rangle)^{*_p} \in \Gamma^{*_p}.$$

$(\forall)$ There are some formula F and some variables φ, ψ satisfying [φ safe $\implies \psi$ safe], $(\forall\varphi\, F) \in \Gamma$, $\psi \notin \mathsf{FV}(\Gamma, \forall\varphi\, F)$ and $\mathcal{T} \vdash \Gamma, F_\varphi(\psi)$. Using the assumption $\forall\varphi\, F \in \mathrm{PBF}(\mathcal{X})$ there are some $G \in \mathrm{PBF}(\mathcal{X})$ and some term u with [φ normal $\implies u$ normal], $\forall\varphi\, F \;\equiv\; \forall\varphi{\leq}u\, G$ and $F_\varphi(\psi) \;\equiv\; \psi \leq u \to G_\varphi(\psi)$.

First assume that φ is safe, then the induction hypothesis and $(\bigvee)$-Exportation lead to some $d, r < \omega$ and some term t with $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p\}$ and

$$\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d,|t|_m\langle\vec{x}\rangle}{r,|t|\langle\vec{x}\rangle}\right. \Gamma, \psi \nleq u, G_\varphi(\psi)\langle\vec{x}\rangle$$

for all $\vec{x} \in \omega$. Hence by $(\forall{\leq})$ $\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d+1,|t|_m\langle\vec{x}\rangle}{r,|t|\langle\vec{x}\rangle}\right. \Gamma\langle\vec{x}\rangle$ for all $\vec{x} \in \omega$, because $\forall\varphi{\leq}u\, G\langle\vec{x}\rangle \;\equiv\; \forall\varphi\, F\langle\vec{x}\rangle \in \Gamma\langle\vec{x}\rangle$.

Now we are in the case that φ is normal, then the induction hypothesis and $(\bigvee)$-Exportation yield some $d, r < \omega$, $r > 0$ and some term t with $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p, \psi\}$ and

$$\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d,|t|_m\langle\vec{x},\mathrm{y}\rangle}{r,|t|\langle\vec{x},\mathrm{y}\rangle}\right. \Gamma, \psi \nleq u, G_\varphi(\psi)\langle\vec{x},\mathrm{y}\rangle \tag{10.4}$$

for all $\vec{x}, \mathrm{y} \in \omega$. Fix $\vec{x} \in \omega$.

u is normal and $\mathsf{nFV}(u) \subset \mathsf{nFV}(\forall\varphi{\leq}u\, G) \subset \{x_0, \ldots, x_p\}$. Let $t' :\equiv \sigma[t]_\psi(u)$. Then $\mathsf{FV}(t') \subset \{x_0, \ldots, x_p\}$ and $t\langle\vec{x},\mathrm{y}\rangle \leq t'\langle\vec{x}\rangle$ for $\mathrm{y} \leq u\langle\vec{x}\rangle$.

Let $\mathrm{y} \leq u\langle\vec{x}\rangle$. With the Equality Lemma (and the Substitution Lemma if ψ is safe) (10.4) leads to $\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d,|t'|_m\langle\vec{x}\rangle}{r,|t'|\langle\vec{x}\rangle}\right. \Gamma, G_\varphi(\underline{\mathrm{y}}), \underline{\mathrm{y}} \nleq u\langle\vec{x}\rangle$. By (Ax1) we obtain

$$\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d,|t'|_m\langle\vec{x}\rangle}{r,|t'|\langle\vec{x}\rangle}\right. \Gamma, G_\varphi(\underline{\mathrm{y}}), \underline{\mathrm{y}} \leq u\langle\vec{x}\rangle,$$

hence by a (Cut) $\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d+1,|t'|_m\langle\vec{x}\rangle}{r,|t'|\langle\vec{x}\rangle}\right. \Gamma, G_\varphi(\underline{\mathrm{y}})\langle\vec{x}\rangle$. Applying $(\bigwedge)$ produces $\mathrm{MC}_n^{\mathrm{p}} \left|\frac{d+2,|t'|_m\langle\vec{x}\rangle}{r,|t'|\langle\vec{x}\rangle}\right. \Gamma\langle\vec{x}\rangle$, because

$$\bigwedge_{\mathrm{y}\le u\langle\vec{\mathrm{x}}\rangle^{\mathbb{N}}} G_\varphi(\underline{\mathrm{y}})\langle\vec{\mathrm{x}}\rangle^{*_p} \equiv (\forall\varphi{\le}u\,G\langle\vec{\mathrm{x}}\rangle)^{*_p} \equiv (\forall\varphi\,F\langle\vec{\mathrm{x}}\rangle)^{*_p} \in \Gamma\langle\vec{\mathrm{x}}\rangle^{*_p}.$$

□

10.3 Extended cut-elimination I

In our examinations of fragments of BPA we need a cut-elimination procedure for the semi-formal systems. Of course we need a procedure which carefully reduces cuts, because otherwise the length of the reduced derivations would grow too fast (i.e., it would grow exponentially).

Therefore, we extend the usual elimination procedure. Let $\mathrm{card}(\Gamma)$ be the number of formulas in Γ.

10.3.1 Extended Elimination Lemma *Let Γ_1 be a finite set of ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{r+1}(\mathcal{X})$-formulas and let A be a ${}^{\mathrm{p}}\Pi^{\mathrm{b}}_{r}(\mathcal{X})$-formula and assume $|\,\mathrm{plh}(G^{*_p})| \le l$ for all $G \in \Gamma_1$. Let Δ, Γ be finite sets of $\mathcal{L}^p_\omega$-formulas. Let $s_1, \ldots, s_p, t$ be terms, let*

$$\exists a_1{\le}s_1 \ldots \exists a_p{\le}s_p\, \exists x{\le}t\, A(a_1, \ldots, a_p, x) \in \Gamma_1$$

and assume that for each $G \in \Gamma_1$ there are terms $u_1, \ldots, u_j$ with

$$G \equiv \exists a_{j+1}{\le}s_{j+1} \ldots \exists a_p{\le}s_p\, \exists x{\le}t\, A(u_1, \ldots, u_j, a_{j+1}, \ldots, a_p, x)$$

[for $j = p$ this means $G \equiv \exists x{\le}t\, A(u_1, \ldots, u_p, x)$]. Let $c := \mathrm{card}(\Gamma_1)$. Then

$${}^{\mathrm{p}}\big|\!\frac{m_0}{r+1,l}\,\Gamma, \Gamma_1^{*_p} \;\;\&\;\; \forall G \in \Gamma_1\; {}^{\mathrm{p}}\big|\!\frac{m_1}{r+1,l}\,\Delta, \neg G^{*_p} \implies {}^{\mathrm{p}}\big|\!\frac{m_0+m_1+c-1}{r+1,l}\,\Gamma, \Delta$$

and

$$\mathrm{MC}^{\mathrm{p}}_n\big|\!\frac{m_0,k}{r+1,l}\,\Gamma, \Gamma_1^{*_p} \;\;\&\;\; \forall G \in \Gamma_1\; \mathrm{MC}^{\mathrm{p}}_n\big|\!\frac{m_1,k}{r+1,l}\,\Delta, \neg G^{*_p}$$
$$\implies \mathrm{MC}^{\mathrm{p}}_n\big|\!\frac{m_0+m_1+c-1,k}{r+1,l}\,\Gamma, \Delta$$

Proof: We use induction on m_0. (We only consider the first assertion, the second one follows in a similar way.) For the rest of the proof we drop the superscript $*_p$. The only interesting case is that the main formula F of the last inference is in Γ_1. Then there are $j \le p$ and some $u_1, \ldots, u_j$ with

$$F \equiv \exists a_{j+1}{\le}s_{j+1} \ldots \exists a_p{\le}s_p\, \exists x{\le}t\, A(u_1, \ldots, u_j, a_{j+1}, \ldots, a_p, x).$$

First assume $j = p$. Then $F \equiv \exists x{\le}t\, A(u_1, \dots, u_p, x)$ and the last inference was $(\bigvee)$, thus there is some $n \le t^{\mathbb{N}}$ and $m' < m_0$ with ${}^{\mathrm{p}}\vert\!\frac{m'}{r+1,l}\, \Gamma, \Gamma_1, A(u_1, \dots, u_p, \underline{n})$. The induction hypothesis yields

$$ {}^{\mathrm{p}}\vert\!\frac{m'+m_1+c-1}{r+1,l}\, \Gamma, \Delta, A(u_1, \dots, u_p, \underline{n}). $$

As $F \in \Gamma_1$ the second assumption yields

$$ {}^{\mathrm{p}}\vert\!\frac{m_1}{r+1,l}\, \Delta, \forall x{\le}t\, \neg A(u_1, \dots, u_p, x), $$

hence ${}^{\mathrm{p}}\vert\!\frac{m_1}{r+1,l}\, \Delta, \neg A(u_1, \dots, u_p, \underline{n})$ by $(\bigwedge)$-Inversion. As $A \in {}^{\mathrm{p}}\Pi^{\mathrm{b}}_r(\mathcal{X})$ we can apply (Cut) to derive the assertion, because $m_1 \le m' + m_1 + c - 1 < m_0 + m_1 + c - 1$.

Now we assume $j < p$. Let

$$ G(b) :\equiv \exists a_{j+2}{\le}s_{j+2} \dots \exists a_p{\le}s_p\, \exists x{\le}t\, A(u_1, \dots, u_j, b, a_{j+2}, \dots, a_p, x), $$

then $F \equiv \exists a_{j+1}{\le}s_{j+1}\, G(a_{j+1})$. The last inference has to be $(\exists{\le})$, thus there is some term v and $m' < m_0$ with

$$ {}^{\mathrm{p}}\vert\!\frac{m'}{r+1,l}\, \Gamma, \Gamma_1, G(v) $$

and $(v \not\le s_{j+1}) \in \Gamma$. Let $\Gamma_2 := \Gamma_1 \cup \{G(v)\}$. Using the second assumption and $F \in \Gamma_1$ we know ${}^{\mathrm{p}}\vert\!\frac{m_1}{r+1,l}\, \Delta, \forall a_{j+1}{\le}s_{j+1}\, \neg G(a_{j+1})$, hence ${}^{\mathrm{p}}\vert\!\frac{m_1}{r+1,l}\, \Delta, \neg G(v), v \not\le s_{j+1}$ by $(\forall{\le})$-Inversion. Let $\Delta' := \Delta \cup \{v \not\le s_{j+1}\}$. We obtain $\forall H \in \Gamma_2\ {}^{\mathrm{p}}\vert\!\frac{m_1}{r+1,l}\, \Delta', \neg H$, thus

$$ {}^{\mathrm{p}}\vert\!\frac{m'+m_1+\mathrm{card}(\Gamma_2)-1}{r+1,l}\, \Delta', \Gamma $$

by the induction hypothesis. Now $\Delta', \Gamma = \Delta, \Gamma$ because $(v \not\le s_{j+1}) \in \Gamma$, hence $m' + m_1 + \mathrm{card}(\Gamma_2) - 1 \le m' + m_1 + (c+1) - 1 \le m_0 + m_1 + c - 1$ yields the assertion. □

10.3.2 Extended Elimination Theorem

$$ {}^{\mathrm{p}}\vert\!\frac{m}{r+1,l}\, \Gamma \implies {}^{\mathrm{p}}\vert\!\frac{2_r(m)}{1,l}\, \Gamma $$

$$ \mathrm{MC}_n^{\mathrm{p}}\vert\!\frac{m,k}{r+n+1,l}\, \Gamma \implies \mathrm{MC}_n^{\mathrm{p}}\vert\!\frac{2_r(m),k}{n+1,l}\, \Gamma $$

Proof: The proof is by induction on m. □

10.4 Extended cut-elimination II

In ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-formulas the impredicative variables serve merely as parameters, thus the ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-formulas can be viewed as $\mathcal{L}_\omega$-formulas in an extended language with additional parameters; the additional graphs $\mathcal{G}_{\mathrm{f}}(\vec{a}, b)$ for $f \in \mathcal{F}^i$ can be taken as $\underline{\mathrm{f}}(\vec{a}) = b$, and as $\underline{\mathrm{f}}$ is primitive recursive all predicative ground instances of this can be viewed as $\mathcal{L}_\omega$-formulas.

From this point of view we can adapt the main parts of the $\mathcal{L}_\omega$-cut-reduction procedure from Chapter 5 to ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-cut-reduction. The definition of the choice sequences $\mathrm{S}(F)$ and the inversions F^f for $f \in \mathrm{S}(F)$ and $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$ directly carries over from that for $\mathcal{L}_\omega$ in Chapter 5. We can transfer the proof of the $\mathcal{L}_\omega$-Inversion word by word and obtain:

10.4.1 Theorem (${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Inversion) *Let* $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$, $f \in \mathrm{S}(F)$ *and* ${}^{\mathrm{p}}\vdash^{m}_{r,l} \Delta, F$, *then* ${}^{\mathrm{p}}\vdash^{m}_{r,l} \Delta, F^f$. □

The $\mathcal{L}_\omega$-Cut Elimination Lemma from Chapter 5 can be rewritten in the form

10.4.2 ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Cut-Elimination Lemma *Let* $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$, $r, l > 0$, ${}^{\mathrm{p}}\vdash^{m}_{r,l} \Delta, F$ *and* ${}^{\mathrm{p}}\vdash^{m}_{r,l} \Delta, \neg F$, *then* ${}^{\mathrm{p}}\vdash^{m+\mathrm{lh}(F)}_{r,l} \Delta$. □

Now we can prove

10.4.3 ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Cut-Elimination Theorem

$$l > 0 \ \& \ {}^{\mathrm{p}}\vdash^{m}_{1,l} \Delta \implies {}^{\mathrm{p}}\vdash^{m \cdot l}_{1,1} \Delta$$

Proof: We use induction on m. The only interesting case, which does not follow immediately (from the induction hypothesis if $m > 0$), is that ${}^{\mathrm{p}}\vdash^{m}_{1,l} \Delta$ is given by a (Cut). In this case there are $m' < m$ and some $\mathcal{L}^p_\omega$-formula F with $\mathrm{prk}(F) = 0$, hence $F \in {}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$, $\mathrm{lh}(F) \leq l$ and ${}^{\mathrm{p}}\vdash^{m'}_{1,l} \Delta, F$ and ${}^{\mathrm{p}}\vdash^{m'}_{1,l} \Delta, \neg F$. The induction hypothesis leads to ${}^{\mathrm{p}}\vdash^{m' \cdot l}_{1,1} \Delta, F$ and ${}^{\mathrm{p}}\vdash^{m' \cdot l}_{1,1} \Delta, \neg F$, thus we obtain by the ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Cut-Elimination Lemma ${}^{\mathrm{p}}\vdash^{m' \cdot l + \mathrm{lh}(F)}_{1,1} \Delta$. We compute $m' \cdot l + \mathrm{lh}(F) \leq m' \cdot l + l \leq m \cdot l$. □

The results of this chapter provide:

10.4.4 Theorem *Let* $F \in \mathrm{PBF}(\mathcal{X})$, $\mathsf{nFV}(F) \subset \{x_1, \ldots, x_p\}$, *and* $n \geq m \geq 0$. *Assume* ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})\text{-L}^m\mathrm{Ind} \vdash F$, *then there is some* $\mathcal{L}_{BPA}$-*term* t *with* $\mathsf{FV}(t) \subset \{x_0, \ldots, x_p\}$ *and some* $c \in \omega$ *such that*

$$\forall \vec{\mathrm{x}} \in \omega \quad \mathrm{dptc}(F, \vec{x})(\vec{\mathrm{x}}) \leq 2_n(c \cdot |t\langle\vec{\mathrm{x}}\rangle|_{m+1}).$$

Proof: The Embedding Theorem 10.2.2 gives us some $\mathcal{L}_{BPA}$-term t and some $d, r < \omega$ with

$$\forall \vec{\mathrm{x}} \in \omega \ \ \mathrm{MC}^{\mathrm{p}}_n \vdash^{d, |t|_m\langle\vec{\mathrm{x}}\rangle}_{r, |t|\langle\vec{\mathrm{x}}\rangle} F\langle\vec{\mathrm{x}}\rangle^{*_p}.$$

Fix $\vec{\mathrm{x}} \in \omega$. The Extended Elimination Theorem shows

$$\mathrm{MC}^{\mathrm{p}}_n \vdash^{2_r(d), |t|_m\langle\vec{\mathrm{x}}\rangle}_{n+1, |t|\langle\vec{\mathrm{x}}\rangle} F\langle\vec{\mathrm{x}}\rangle^{*_p},$$

thus Theorem 10.2.1 and the observation $|t+1|_m > 0$ yield

$$^{\mathrm{p}}\vdash^{2_r(d) \cdot |t+1|_{m+1}\langle\vec{\mathrm{x}}\rangle}_{n+1, |t|\langle\vec{\mathrm{x}}\rangle} F\langle\vec{\mathrm{x}}\rangle^{*_p}.$$

Now the Extended Elimination Theorem yields

$$^{\mathrm{p}}\vdash^{2_n[2_r(d) \cdot |t+1|_{m+1}\langle\vec{\mathrm{x}}\rangle]}_{1, |t|\langle\vec{\mathrm{x}}\rangle} F\langle\vec{\mathrm{x}}\rangle^{*_p},$$

hence

$$^{\mathrm{p}}\vdash^{2_n[2_r(d) \cdot |t+1|_{m+1}\langle\vec{\mathrm{x}}\rangle] \cdot |t|\langle\vec{\mathrm{x}}\rangle}_{1,1} F\langle\vec{\mathrm{x}}\rangle^{*_p}$$

by the ${}^{\mathrm{p}}\Delta^{\mathrm{b}}_0(\mathcal{X})^{*_p}$-Cut-Elimination Theorem. Let $s :\equiv t\langle\vec{\mathrm{x}}\rangle +$ "some constant" so that $|s|_{m+1} \geq 2$ for all $\vec{\mathrm{x}} \in \omega$. Let $c := 2_r(d)$. In our following estimations we use $2^{\mathrm{x}} \cdot \mathrm{y} \leq 2^{\mathrm{x} \cdot \mathrm{y}}$ and $\mathrm{x} + \mathrm{y} \leq \mathrm{x} \cdot \mathrm{y}$ for $\mathrm{x}, \mathrm{y} \geq 2$, and $\mathrm{x} < 2^{|\mathrm{x}|}$. Let $n' := n - m \geq 0$. If $m > 0$ we compute

$$\begin{aligned} \mathrm{dptc}(F, \vec{x})(\vec{\mathrm{x}}) &\leq 2_{n'+m}(c \cdot |s|_{m+1}) \cdot |s| \\ &\leq 2_{n'}(2_m(c \cdot |s|_{m+1}) \cdot |s|) \\ &\leq 2_n((c+1) \cdot |s|_{m+1}). \end{aligned}$$

In the case $m = 0$ we have

$$\begin{aligned} \mathrm{dptc}(F, \vec{x})(\vec{\mathrm{x}}) &\leq 2_n(c \cdot |s|) \cdot |s| \\ &\leq 2_n(c \cdot |s| \cdot |s|) \\ &\leq 2_n(c \cdot |s \# s|). \end{aligned}$$

□

Chapter 11

Predicative Boundedness Theorems (PBT)

11.1 Preliminaries

One of the main tools in the proof-theoretical investigation of $I\Sigma_n^0$ is the boundedness theorem 5.2.5. Here we need predicative versions of it:

11.1.1 Predicative Boundedness Theorem

$$ {}^{\mathrm{p}}\!\!\mid\!\frac{m}{1,1}\, BigFun(a,b,\alpha,X) \implies \Phi_{\mathcal{E}}(\alpha) \leq m $$

and

11.1.2 Predicative Boundedness Theorem

$$ {}^{\mathrm{p}}\!\!\mid\!\frac{m}{1,1}\, BigFun(a,b,\alpha,X(d)) \implies \Phi_{\mathcal{E}}(\alpha) \leq m. $$

Remember that $X(d) :\equiv \{\varphi : \mathrm{Bit}(\varphi,d)\}$. Of course the first theorem follows from the second one because we can show

$$ {}^{\mathrm{p}}\!\!\mid\!\frac{m}{1,1}\, \Delta \implies {}^{\mathrm{p}}\!\!\mid\!\frac{m}{1,1}\, \Delta_X(\{a : A(a)\}) $$

for any atomic formula $A(a)$. The first Predicative Boundedness Theorem 11.1.1 can also be obtained by adapting the Boundedness Theorem 5.2.5 to $\textsc{bsf}^{\mathrm{p}}$:

> Let $\bar{*}$: (ground $\mathcal{L}_\omega^p$-formulas) $\to$ $\mathcal{L}_\omega$ be homeomorphic up to $(\forall a{\leq}s\, F(a))^{\bar{*}} :\equiv \bigwedge_{k\leq s^{\mathbb{N}}} F(\underline{k})^{\bar{*}}$ and $(\exists a{\leq}s\, F(a))^{\bar{*}} :\equiv \bigvee_{k\leq s^{\mathbb{N}}} F(\underline{k})^{\bar{*}}$ and defined to be the identity on the atomic $\mathcal{L}_\omega^p$-formulas.

Then we obtain for any finite set Δ of ground $\mathcal{L}_\omega^p$-formulas

$$^{\mathrm{p}}\!\mathop{\models}\limits^{m}_{1,1} \Delta \implies \mathop{\models}\limits^{m}_{1} \Delta^{\bar{*}}.$$

Thus, we can apply, in essential, the Boundedness Theorem 5.2.5.

The proof of the second Predicative Boundedness Theorem 11.1.2 is a nontrivial modification of the Boundedness Theorem 5.2.5. The proof of the latter essentially uses the monotonicity of formulas F in which X occurs only positively, i.e., not in the form $s \notin X$:

$$M \subset N \;\&\; \mathbb{N} \models F_X[M] \implies \mathbb{N} \models F_X[N].$$

If we replace X with the set $X(d)$ coded by d then we want to obtain something like

$$m \subset n \text{ , i.e., } \forall i\,(\mathrm{Bit}(i,m) \to \mathrm{Bit}(i,n)), \;\&\; \mathbb{N} \models F_d[m] \implies \mathbb{N} \models F_d[n]$$

for formulas F in which $\mathrm{Bit}^{\mathrm{c}}(\cdot, d)$ does not occur. But then we have the problem that d can also occur in terms and atomic formulas other than Bit. Therefore, we first have to find a notion of sets of indiscernibles $I \subset \omega$ to a given set Π of formulas and $l \in \omega$, which provides

$$\forall M \subset \{0, \dots, l\}\; \exists m{\in}I\, (m \text{ codes } M \text{ below } l),$$

at which a number m codes a set M below l iff $\forall i{\le}l\,(i{\in}M \leftrightarrow \mathrm{Bit}(i,m))$, and

$$\exists m \in I\; (\mathbb{N} \models A_d[m]) \iff \forall m \in I\; (\mathbb{N} \models A_d[m])$$

for any atomic formula $A \in \Pi$ other than $\mathrm{Bit}(\cdot, d)$ or $\mathrm{Bit}^{\mathrm{c}}(\cdot, d)$.

11.2 Indiscernibles

We first characterize the sets of formulas for which we want to find indiscernibles.

11.2.1 Definition *Let* $\mathrm{GB}(l)$ *be the set of all predicative ground terms* t *and* $\mathcal{L}_\omega^p$*-formulas* F *such that every ground term* s *which occurs in* t *resp.* F *satisfies* $s^{\mathbb{N}} \le l$*. A predicative ground formula* $F \in \mathrm{PBF}(\mathcal{X})$ *is in* $\mathrm{GB}(l)$ *iff* $F^{*_p} \in \mathrm{GB}(l)$*.*

11.2.2 Lemma *Let $F \in \textsc{gb}(l)$.*

1. $l \leq m \implies F \in \textsc{gb}(m)$.

2. $k \leq l \implies F_a(\underline{k}) \in \textsc{gb}(l)$.

Proof: The proof of 1. is obvious. 2. follows easily by induction on the generation of F from the following observation for terms s:

$$\begin{aligned} & s \in \textsc{gb}(l) \\ & \implies s \text{ is a ground term with } s^{\mathbb{N}} \leq l \text{ or } s \text{ is a safe variable} \\ & \implies s_a(\underline{k}) \in \textsc{gb}(l). \end{aligned}$$

□

Now we define indiscernibles for formulas in $\textsc{gb}(l)$. Remember that $\mathrm{Bit}(k, n)$ holds iff the k-th bit in the binary expansion of n is 1. We define the set of *l-indiscernibles* $\textsc{Isc}_l$ and related things where we identify $(l+1)$ with its usual set theoretical representation $\{0, \ldots, l\}$.

$$\begin{aligned} h(l) &:= \max\Big(\{\Phi_{\mathcal{E}}(\alpha) \,:\, \alpha \in \mathcal{E} \cap (l+1)\} \cup \{l\}\Big) \\ \textsc{Isc}_l &:= \{n \in \omega \,:\, \forall k{>}l\, [\mathrm{Bit}(k,n) \leftrightarrow k = 2^{h(l)} \textit{ or } k = 2^{h(l)} + 3]\}, \\ M_l(n) &:= \{k \leq l \,:\, \mathrm{Bit}(k,n)\}, \\ Z_l(X) &:= 2^{2^{h(l)}+3} + 2^{2^{h(l)}} + \textstyle\sum_{i \in X \cap (l+1)} 2^i \end{aligned}$$

$$d_0 \sqsubseteq^l d_1 \;:\!\iff d_0, d_1 \in \textsc{Isc}_l \textit{ and } M_l(d_0) \subset M_l(d_1).$$

l-Indiscernibles and the relation $\sqsubseteq^l$

Observe that

$$\begin{array}{lcl} M_l(Z_l(X)) & = & X \cap (l+1), \\ Z_l(M_l(n)) & = & n \quad for\ n \in \textsc{Isc}_l, \\ Z_l[\mathfrak{P}(l+1)] & = & \textsc{Isc}_l, \\ M_l[\textsc{Isc}_l] & = & \mathfrak{P}(l+1). \end{array}$$

The crucial point is to observe that $\textsc{Isc}_l$ *is* a set of indiscernibles for the set of atomic formulas in $\textsc{gb}(l)$ without Bit and Bit^c. In essential, this is true because of three reasons:

1. all functions in $\mathcal{F}^i$ are polytime functions and thus have polynomial growth rate,

2. the l-indiscernibles are "very much" bigger than the values of the ground terms t that could occur in a formula in $\textsc{gb}(l)$, i.e., $t^{\mathbb{N}} \leq l$ and $\forall x \in \textsc{Isc}_l(2^{2^l} < x)$,

3. no l-indiscernibles is a sequence-number, because the highest bits of an indiscernible are always of the form $10010\ldots$ which cannot be the highest bits of a sequence-number: sequence-numbers are build up from $00, 10, 11$, thus 1001 and 010010 are impossible beginnings.

11.2.3 Main Lemma *Let $F \in \textsc{gb}(l)$ be an atomic formula other than* Bit$(\cdot, d)$ *or* Bit$^c(\cdot, d)$ *with* $\mathsf{FV}(F) \subset \{d\}$, *then*

$$\forall d \in \textsc{Isc}_l\ \mathbb{N} \models F \quad or \quad \forall d \in \textsc{Isc}_l\ \mathbb{N} \not\models F.$$

Proof: We postpone this to the Appendix C. □

11.3 Negative points and monotonicity

11.3.1 Definition *Let $\textsc{qb}(\vec{a})$ be the set of all $\mathcal{L}^p_\omega$-formulas F such that every quantifier which occurs in F is bounded by a variable from the list $\vec{a}$ or by some ground $\mathcal{L}_{BPA}$-term.*

We define the negative points of a formula as in Chapter 5.

11.3.2 Definition *The* negative points $\mathrm{N}_d(F) \subset \omega$ *of a formula* $F \in \textsc{qb}(\emptyset)$ *with* $\mathsf{FV}(F) \subset \{d\}$ *relative to the safe variable* d *are defined by the following clauses:*

1. *If* F *is atomic let*
$$\mathrm{N}_d(F) := \begin{cases} \{s^{\mathbb{N}}\} : \textit{if } F \equiv \mathrm{Bit}^{\mathrm{c}}(s,d) \textit{ and } s \not\equiv d \\ \emptyset \quad\; : \textit{otherwise} \end{cases}$$
2. $\mathrm{N}_d\left(\bigvee_{i\leq n} F_i\right) := \mathrm{N}_d\left(\bigwedge_{i\leq n} F_i\right) := \bigcup_{i\leq n} \mathrm{N}_d(F_i)$
3. $\mathrm{N}_d\left(\forall a{\leq}s\, F(a)\right) := \mathrm{N}_d\left(\exists a{\leq}s\, F(a)\right) := \bigcup_{l\leq s^{\mathbb{N}}} \mathrm{N}_d(F(\underline{l}))$

For sets of $\mathcal{L}^p_\omega$*-formulas* Δ *we define* $\mathrm{N}_d(\Delta) := \bigcup_{F\in\Delta} \mathrm{N}_d(F)$.

Case 1. of this definition is well-defined because if $s \not\equiv d$ then s has to be a ground term. Case 3. is well-defined because $\mathrm{Q}a{\leq}s\, F(a) \in \textsc{qb}(\emptyset)$, thus s has to be a ground term.

11.3.3 Lemma (Monotonicity) *Let* $F \in \textsc{gb}(l)$ *and* $F \in \textsc{qb}(\emptyset)$ *with* $\mathsf{FV}(F) \subset \{d\}$. *Assume* $d_0 \sqsubseteq^l d_1$ *with* $\mathrm{N}_d(F) \subset M_l(d_0)$, *then*

$$\mathbb{N} \models F_d[d_0] \implies \mathbb{N} \models F_d[d_1].$$

Proof: The proof is by induction on the generation of F. First we observe that

$$u_d[e]^{\mathbb{N}} \leq e$$

for $u \in \textsc{gb}(l)$ with $\mathsf{FV}(u) \subset \{d\}$ and $e \in \textsc{Isc}_l$. For if $u \not\equiv d$ then u is ground, thus $u \leq l$ as $u \in \textsc{gb}(l)$, and $l < e$ by definition of $\textsc{Isc}_l$.

If F is atomic we distinguish the following cases:

1. $F \not\equiv \mathrm{Bit}(u,v)$ and $F \not\equiv \mathrm{Bit}^{\mathrm{c}}(u,v)$. $F \in \textsc{gb}(l)$ and $d_0, d_1 \in \textsc{Isc}_l$, thus d_0, d_1 are indiscernibles for F, i.e., Lemma 11.2.3 implies
$$\mathbb{N} \models F_d[d_0] \iff \mathbb{N} \models F_d[d_1].$$
2. $F \equiv \mathrm{Bit}(u,v)$. If d does not occur in F the assertion is obvious. Otherwise, assume $\mathbb{N} \models \mathrm{Bit}(u,v)_d[d_0]$. Then
$$u_d[d_0]^{\mathbb{N}} < |v_d[d_0]|^{\mathbb{N}} \leq |d_0|^{\mathbb{N}},$$
hence $u \not\equiv d$. But then u has to be a ground term and $v \equiv d$. Now $\mathbb{N} \models F_d[d_0]$ together with the assumptions yields
$$u^{\mathbb{N}} \in M_l(d_0) \subset M_l(d_1),$$
hence $\mathbb{N} \models \mathrm{Bit}(u,v)_d[d_1]$.

3. $F \equiv \mathrm{Bit}^c(u,v)$. If d does not occur in F the assertion is obvious. If $u \equiv d$ then $\mathbb{N} \models \mathrm{Bit}^c(u,v)_d[d_1]$ for $|v_d[d_1]|^{\mathbb{N}} \leq |d_1|^{\mathbb{N}} < d_1$. Otherwise, u has to be a ground term and $v \equiv d$. By assumptions we have $\mathrm{N}_d(F) = \{u^{\mathbb{N}}\} \subset M_l(d_0)$, hence $\mathbb{N} \models \mathrm{Bit}(u, \underline{d_0})$ which shows $\mathbb{N} \not\models F_d[d_0]$.

If $F \equiv \bigwedge_{i \leq n} F_i$ and $\mathbb{N} \models F_d[d_0]$, then we have $\mathbb{N} \models (F_i)_d[d_0]$, thus $\mathbb{N} \models (F_i)_d[d_1]$ for any $i \leq n$ by the induction hypothesis. Hence $\mathbb{N} \models F_d[d_1]$.

If $F \equiv \bigvee_{i \leq n} F_i$ and $\mathbb{N} \models F_d[d_0]$, then we have $\mathbb{N} \models (F_i)_d[d_0]$. Therefore we obtain $\mathbb{N} \models (F_i)_d[d_1]$ for some $i \leq n$ by the induction hypothesis. Hence $\mathbb{N} \models F_d[d_1]$.

If $F \equiv \forall a{\leq}s\, G$, then s has to be a ground term, for $F \in \mathrm{QB}(\emptyset)$. Thus $F \in \mathrm{GB}(l)$ implies $s^{\mathbb{N}} \leq l$. Lemma 11.2.2 yields that $G_a(\underline{k}) \in \mathrm{GB}(l)$, obviously also $G_a(\underline{k}) \in \mathrm{QB}(\emptyset)$, for any $k \leq s^{\mathbb{N}}$. Assume $\mathbb{N} \models F_d[d_0]$, then we have $\mathbb{N} \models (G_a(\underline{k}))_d[d_0]$, hence $\mathbb{N} \models (G_a(\underline{k}))_d[d_1]$ by the induction hypothesis, for any $k \leq s^{\mathbb{N}}$. Hence $\mathbb{N} \models F_d[d_1]$.

If $F \equiv \exists a{\leq}s\, G$, then s has to be a ground term, because $F \in \mathrm{QB}(\emptyset)$. Now $F \in \mathrm{GB}(l)$ implies $s^{\mathbb{N}} \leq l$. Using Lemma 11.2.2 we obtain that $G_a(\underline{k}) \in \mathrm{GB}(l)$, obviously also $G_a(\underline{k}) \in \mathrm{QB}(\emptyset)$, for any $k \leq s^{\mathbb{N}}$. Assume $\mathbb{N} \models F_d[d_0]$, then we have $\mathbb{N} \models (G_a(\underline{k}))_d[d_0]$, hence $\mathbb{N} \models (G_a(\underline{k}))_d[d_1]$ by the induction hypothesis, for some $k \leq s^{\mathbb{N}}$. Hence $\mathbb{N} \models F_d[d_1]$. □

11.4 Proving PBT

We adapt the definition of the reachability operator for orderings from Chapter 5 to $\prec$, the fixed ordering of the exponential codes $\mathcal{E}$. For $N \subset \omega$ let

$$\mathrm{R}^m(N) := \{e \in \omega \;:\; e \notin \mathcal{E} \text{ or } \Phi_{\mathcal{E}}(e) \leq \overline{\mathrm{en}}_N(m)\} \cup N$$

and

$$\mathrm{R}^m_l(N) := Z_l(\mathrm{R}^m(N)) = 2^{2^{h(l)+3}} + 2^{2^{h(l)}} + \sum_{i \in \mathrm{R}^m(N) \cap (l+1)} 2^i$$

Remember that $\overline{\mathrm{en}}_N$ is the dual enumeration function $\mathrm{en}_{\mathrm{ON} \setminus N}$ from Chapter 5 which in this context (where we consider only finite ordinals) can be written as $\mathrm{en}_{\omega\setminus N}$. Analogously to Chapter 5 we observe

$$N \subset N' \implies \overline{\mathrm{en}}_N(m) \leq \overline{\mathrm{en}}_{N'}(m) \tag{11.1}$$

$$\overline{\mathrm{en}}_{N\cup\{e\}}(m) \leq \overline{\mathrm{en}}_N(m+1) \tag{11.2}$$

$$\mathrm{R}^m(N \cup \{e\}) \subset \mathrm{R}^{m+1}(N) \cup \{e\} \tag{11.3}$$

$$m \leq m' \ \& \ N \subset N' \implies \mathrm{R}^m_l(N) \sqsubseteq^l \mathrm{R}^{m'}_l(N') \tag{11.4}$$

If $\mathrm{A}(N)$ denotes the *accessibility operator*, i.e.,

$$\mathrm{A}(N) \ := \ N \cup \{k \in \omega \ : \ \forall l \prec k(l \in N)\},$$

and $\mathrm{A}^m(N)$ its iterations, i.e., $\mathrm{A}^m(N) \ := \ \mathrm{A}(N \cup \bigcup_{n<m} \mathrm{A}^n(N))$, then

$$\mathrm{R}^m(N) = \mathrm{A}^m(N) \tag{11.5}$$

for all $m \in \omega$.

Let ${}^{\mathrm{p}}_{l}\!\mid\!\frac{m}{1,1}\, \Delta$ be the restriction of ${}^{\mathrm{p}}\!\mid\!\frac{m}{1,1}$ to derivations where all occurring terms are in $\mathrm{GB}(l)$. We obtain

$$^{\mathrm{p}}\!\mid\!\frac{m}{1,1}\, \Delta \implies \exists l < \omega \ \ {}^{\mathrm{p}}_{l}\!\mid\!\frac{m}{1,1}\, \Delta \tag{11.6}$$

because the derivation trees are finite. Let $X(d) \ :\equiv \ \{\varphi \ : \ \mathrm{Bit}(\varphi, d)\}$.

11.4.1 Predicative Boundedness Lemma *Suppose* $\mathbb{N} \models Big(\underline{\mathrm{a}}, \underline{l}, \alpha)$ *with* $\mathrm{a}, l \in \omega$ *and* α *a ground term, and*

$${}^{\mathrm{p}}_{l}\!\mid\!\frac{m}{1,1}\, \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta$$

with $\mathsf{FV}(\Delta) \subset \{\vec{c}, d\}$ *and* $\Delta \in \mathrm{QB}(\emptyset)$. *Then*

$$\forall \vec{\mathrm{c}} \leq l \quad \mathbb{N} \models \Delta_{\vec{c},d}[\vec{\mathrm{c}}, \mathrm{R}^m_l(\mathrm{N}_d(\Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})))]. \tag{11.7}$$

Proof: We use induction on m. In the sequel we use validity in the standard model $\mathbb{N}$ sloppily, e.g. we write $s_d[n] \prec \alpha$ instead of $\mathbb{N} \models (s \prec \alpha)_d[n]$ etc.

We distinguish several cases concerning the last inference. If this is an axiom then Δ has to be the same axiom and (11.7) follows by the validity of the axioms. The cases of a $(\bigwedge)$ or $(\bigvee)$-inference follow directly from the induction hypothesis, the Monotonicity Lemma 11.3.3, observation (11.4) and the correctness of the inferences $(\bigwedge)$ resp. $(\bigvee)$.

In the case that the last inference is $(\forall\leq)$ there are $m' < m$, some safe variables e, f and some term $s \in \mathrm{GB}(l)$ such that $(\forall e{\leq}s\, F) \in \Delta$, $f \notin \mathsf{FV}(\Delta) \cup \{d\}$ and

$$ {}^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, F_e(f), f \not\leq s. $$

By assumption $(\forall e{\leq}s\, F) \in \mathrm{QB}(\emptyset)$, thus $F_e(f) \in \mathrm{QB}(\emptyset)$ and s is a ground term with $s^{\mathbb{N}} \leq l$. Applying the induction hypothesis and the Monotonicity Lemma 11.3.3 we obtain, as $\mathrm{N}_d(F_{\vec{c},e}(\underline{\vec{\mathrm{c}}}, \underline{k})) \subset \mathrm{N}_d(\Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}}))$ for $k \leq s^{\mathbb{N}}$,

$$ \forall \vec{\mathrm{c}}{\leq}l\, \forall k{\leq}s^{\mathbb{N}} \;\; \mathbb{N} \models \Delta_{\vec{c},d}[\vec{\mathrm{c}}, \mathrm{R}^m_l(\mathrm{N}_d(\Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})))], F_{\vec{c},e,d}[\vec{\mathrm{c}}, k, \mathrm{R}^m_l(\mathrm{N}_d(\Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})))], $$

hence

$$ \forall \vec{\mathrm{c}}{\leq}l \;\; \mathbb{N} \models \Delta_{\vec{c},d}[\vec{\mathrm{c}}, \mathrm{R}^m_l(\mathrm{N}_d(\Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})))]. $$

If the last inference is $(\exists\leq)$ and $\neg Prog(\underline{\mathrm{a}}, \alpha, X(d))$ is not its main formula then a similar argument as for $(\forall\leq)$ yields the assertion (11.7).

If $\neg Prog(\underline{\mathrm{a}}, \alpha, X(d))$ is the main formula then there are a term $s \in \mathrm{GB}(l)$ and $m' < m$ such that

$$ {}^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, s \prec \alpha \wedge (s \sqsubset X(d))^{\underline{\mathrm{a}}} \wedge s \notin X(d) $$

and $(s \not\leq \underline{\mathrm{a}}) \in \Delta$. Applying $(\bigwedge)$-Inversion yields

$$ {}^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, s \prec \alpha \tag{11.8} $$

$$ {}^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, (s \sqsubset X(d))^{\underline{\mathrm{a}}} \tag{11.9} $$

$$ {}^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, \mathrm{Bit}^{\mathrm{c}}(s, d). \tag{11.10} $$

We may assume that $\mathsf{FV}(s) \subset \{\vec{c}, d\}$. Fix some $\vec{c} \leq l$ and let $s' :\equiv s_{\vec{c}}(\underline{\vec{\mathrm{c}}})$ and $\Delta' := \Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})$. Observe that the formulas in (11.8) to (11.10) are in $\mathrm{QB}(\emptyset)$. We compute $\mathrm{N}_d(s \prec \alpha) = \mathrm{N}_d((s \sqsubset X(d))^{\underline{\mathrm{a}}}) = \emptyset$.

If $s'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))] \not\prec \alpha$ then the induction hypothesis applied to (11.8) yields $\mathbb{N} \models \Delta'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))]$. The Monotonicity Lemma 11.3.3 and (11.4) imply the assertion (11.7).

Otherwise, $s'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))] \prec \alpha$, thus $s \not\equiv d$, and s' has to be a ground term. Then $s' \leq l$, because $s \in \mathrm{GB}(l)$ is a ground term, or s is some c_i and $\vec{c} \leq l$ otherwise. If there is some $k \prec s'$ with $k \notin \mathrm{R}^{m'}(\mathrm{N}_d(\Delta'))$, then the induction hypothesis applied to (11.9) yields $\mathbb{N} \models \Delta'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))]$ for $k \prec s' \prec \alpha$, and $Big(\mathrm{a}, l, \alpha)$ implies $k < \mathrm{a}$.

Again the Monotonicity Lemma 11.3.3 and (11.4) yield the assertion (11.7).

If $k \in \mathrm{R}^{m'}(\mathrm{N}_d(\Delta'))$ for all $k \prec s'$, then (11.5) yields

$$s'^{\mathbb{N}} \in \mathrm{R}^{m'+1}(\mathrm{N}_d(\Delta')). \tag{11.11}$$

We compute $\mathrm{N}_d(\mathrm{Bit^c}(s', d)) = \{s'^{\mathbb{N}}\}$, hence

$$\mathbb{N} \vDash \Delta'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\})]$$

by the induction hypothesis applied to (11.10). With (11.3) and (11.11) we compute

$$\begin{aligned} \mathrm{R}^{m'}(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\}) &\subset \mathrm{R}^{m'+1}(\mathrm{N}_d(\Delta')) \cup \{s'^{\mathbb{N}}\} \\ &\subset \mathrm{R}^{m}(\mathrm{N}_d(\Delta')), \end{aligned}$$

hence

$$\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\}) \sqsubseteq^l \mathrm{R}^{m}_l(\mathrm{N}_d(\Delta')),$$

and the claim follows with the Monotonicity Lemma 11.3.3.

In the case that the last inference is a (Cut) there are some $m' < m$ and some atomic formula F such that ${}^{\mathrm{P}}_{l}\!\left|\frac{m'}{1,1}\right. \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, F$ and ${}^{\mathrm{P}}_{l}\!\left|\frac{m'}{1,1}\right. \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \Delta, \neg F$. We may assume $\mathsf{FV}(F) \subset \{\vec{c}, d\}$. Fix some $\vec{\mathrm{c}} \leq l$ and let $F' :\equiv F_{\vec{c}}(\underline{\vec{\mathrm{c}}})$ and $\Delta' := \Delta_{\vec{c}}(\underline{\vec{\mathrm{c}}})$. As F is atomic it trivially is in $\mathrm{QB}(\emptyset)$, hence

$$\mathbb{N} \vDash (\Delta', F')_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta', F'))] \tag{11.12}$$

$$\mathbb{N} \vDash (\Delta', \neg F')_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta', \neg F'))] \tag{11.13}$$

by the induction hypothesis. If $F \not\equiv \mathrm{Bit}(s, d)$ and $F \not\equiv \mathrm{Bit^c}(s, d)$ for all terms s, then $\mathrm{N}_d(F') = \mathrm{N}_d(\neg F') = \emptyset$ and the assertion (11.7) follows by the Monotonicity Lemma 11.3.3 and the law of the excluded middle.

Otherwise, we may assume without loss of generality $F \equiv \mathrm{Bit}(s, d)$ for some term s. Then $\mathrm{N}_d(F') = \emptyset$ and $\mathrm{N}_d(\neg F') = \{s'^{\mathbb{N}}\}$ resp. $\mathrm{N}_d(\neg F') = \emptyset$ if $s \equiv d$. If $\mathbb{N} \vDash \neg F'_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))]$ then (11.12) and the Monotonicity Lemma 11.3.3 yields the assertion (11.7). Otherwise, $\mathbb{N} \vDash \mathrm{Bit}(s', d)_d[\mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))]$, hence $s \not\equiv d$ and $s'^{\mathbb{N}} \leq l$, because $F \in \mathrm{GB}(l)$, hence $s'^{\mathbb{N}} \in \mathrm{R}^{m'}_l(\mathrm{N}_d(\Delta'))$. This and (11.3) lead to

$$\begin{aligned} \mathrm{R}^{m'}(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\}) &\subset \mathrm{R}^{m'+1}(\mathrm{N}_d(\Delta')) \cup \{s'^{\mathbb{N}}\} \\ &\subset \mathrm{R}^{m}(\mathrm{N}_d(\Delta')), \end{aligned}$$

hence

$$\mathrm{R}_l^{m'}(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\}) \sqsubseteq^l \mathrm{R}_l^m(\mathrm{N}_d(\Delta')). \tag{11.14}$$

Now (11.13) yields $\mathbb{N} \models \Delta'_d[\mathrm{R}_l^{m'}(\mathrm{N}_d(\Delta') \cup \{s'^{\mathbb{N}}\})]$, thus (11.14) and the Monotonicity Lemma 11.3.3 produce the assertion (11.7). □

Proof of the Predicative Boundedness Theorem 11.1.2:
Assume $^{\mathrm{p}}\!\vdash^{m}_{1,1} BigFun(a, b, \alpha, X(d))$. Lemma 9.1.1 shows that there are some $\mathrm{b} \geq \mathrm{a}$ such that $\mathbb{N} \models Big(\underline{\mathrm{a}}, \underline{\mathrm{b}}, \alpha)$. Thus, there are $m' < m$ and by (11.6) $l < \omega$, $l \geq \mathrm{b}$, such that

$$^{\mathrm{p}}_{l}\!\vdash^{m'}_{1,1} \neg Prog(\underline{\mathrm{a}}, \alpha, X(d)), \neg Big(\underline{\mathrm{a}}, \underline{\mathrm{b}}, \alpha), (\alpha \sqsubset X(d))^{\underline{\mathrm{a}}}.$$

An inspection of the formulas $\neg Big(\underline{\mathrm{a}}, \underline{\mathrm{b}}, \alpha)$ and $(\alpha \sqsubset X(d))^{\underline{\mathrm{a}}}$ shows that they are in $\textsc{qb}(\emptyset)$ and that $\mathrm{N}_d(\neg Big(\underline{\mathrm{a}}, \underline{\mathrm{b}}, \alpha), (\alpha \sqsubset X(d))^{\underline{\mathrm{a}}}) = \emptyset$. Thus the Predicative Boundedness Lemma 11.4.1 produces

$$\mathbb{N} \models \neg Big(\underline{\mathrm{a}}, \underline{\mathrm{b}}, \alpha), [(\alpha \sqsubset X(d))^{\underline{\mathrm{a}}}]_d(\mathrm{R}_l^{m'}(\emptyset)).$$

Now $Big(\mathrm{a}, \mathrm{b}, \alpha)$ yields $\forall \beta \prec \alpha (\beta \leq \mathrm{a})$, hence

$$\begin{aligned} \forall \beta \prec \alpha (\beta \in \mathrm{R}^{m'}(\emptyset)) \quad &\Longrightarrow \quad \forall \beta \prec \alpha \;\; \Phi_{\mathcal{E}}(\beta) \leq \overline{\mathrm{en}}_{\emptyset}(m') = m' \\ &\Longrightarrow \quad \Phi_{\mathcal{E}}(\alpha) \leq m. \end{aligned}$$

□

Chapter 12

Dynamic Ordinal Analysis (DOA)

With the ordinal analysis of an arithmetical theory T we associate the computation of the proof-theoretical ordinal $\mathcal{O}(T)$ of T, i.e., the supremum of the order-types of the provable well-orderings of T.[1] Usually this yields a *good measurement* of T in the sense that the different theories under consideration receive different proof-theoretical ordinals. For weak theories, i.e., sub-theories of $\mathrm{I}\Sigma^0_1$, R. Sommer showed in his PhD-thesis [20] that

$$\mathrm{I}\Delta^0_0 + \mathit{Fund}(\omega^2, \Delta^0_0) = \mathrm{I}\Sigma^0_1$$

and

$$\mathrm{I}\Delta^0_0 \vdash \mathit{Fund}(\omega \cdot k, \Delta^0_0) \quad \text{for all } k \in \omega.$$

Furthermore, he remarked in [21]

$$\mathrm{S}^1_2(\mathcal{X}) + \mathit{Fund}(\omega^2, \Delta^0_0) = \mathrm{I}\Sigma^0_1$$

and

$$\mathrm{T}^1_2(\mathcal{X}) \vdash \mathit{Fund}(\omega \cdot k, \Delta^0_0) \quad \text{for all } k \in \omega.$$

Therefore we obtain

$$\mathcal{O}(T) = \omega^2$$

for theories T which are stronger than $\mathrm{T}^1_2(\mathcal{X})$ but weaker than $\mathrm{I}\Sigma^0_1$. Thus, the usual ordinal analysis does not yield a good measurement of subsystems of $\mathrm{I}\Sigma^0_1$. In the following we introduce the Dynamic Ordinal

[1] Cf. Chapter 1.

analysis for the theories ${}^{\mathrm{p}}\mathrm{R}_2^n$, ${}^{\mathrm{p}}\mathrm{S}_2^n$, ${}^{\mathrm{p}}\mathrm{T}_2^n$, ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{R}_2^n(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{S}_2^n(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{T}_2^n(\mathcal{X})$, ${}^{\mathrm{p}}\Sigma_n^{\mathrm{b}}(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$. With the Dynamic Ordinal analysis we will overcome the deficiency described above.

12.1 Dynamic Ordinals and separation

For $f, g \in {}^{\omega}\omega$ we define $f \leq g$ iff f is *majorized by* g, i.e.,

$$\forall n\, (f(n) \leq g(n)).$$

For $F \subset {}^{\omega}\omega$ let $\mathcal{H}(F)$ be *the $\leq$-hull of F*:

$$\mathcal{H}(F) \;:=\; \{f \in {}^{\omega}\omega \,:\, \exists g{\in}F\, (f \leq g)\}.$$

12.1.1 Definition *Let T be a theory formulated in $\mathcal{L}_{BPA}(\mathcal{X})$. We define the* Dynamic Ordinal of T *by*

$$\begin{aligned}\mathcal{DO}(T) \;:=\; \mathcal{H}\Big(\{\lambda n.\Phi_{\mathcal{E}}(t(n)) \mid\ & t(x) \textit{ is an } \mathcal{L}_{BPA}\textit{-term with}\\ & \mathsf{FV}\,(t) \subset \{x\} \textit{ such that } \mathbb{N} \vDash \forall x\, (t(x) \in \mathcal{E})\\ & \textit{and } T \vdash \forall x\, BigFun(a, b, t, X)\}\Big).\end{aligned}$$

12.1.2 Definition *Let T be a theory formulated in $\mathcal{L}_{BPA}$. We define the* Dynamic Ordinal of T *by*

$$\begin{aligned}\mathcal{DO}(T) \;:=\; \mathcal{H}\Big(\{\lambda n.\Phi_{\mathcal{E}}(t(n)) \mid\ & t(x) \textit{ is an } \mathcal{L}_{BPA}\textit{-term with}\\ & \mathsf{FV}\,(t) \subset \{x\} \textit{ such that } \mathbb{N} \vDash \forall x\, (t(x) \in \mathcal{E})\\ & \textit{and } T \vdash \forall x\, BigFun(a, b, t, X(d))\}\Big).\end{aligned}$$

With the *Dynamic Ordinal analysis* of a theory T we associate the computation of the Dynamic Ordinal of T. If the Dynamic Ordinal analysis of theories T_1, T_2 yields an inequality between the Dynamic Ordinals of T_1 resp. T_2 then we obtain a separation of T_1 and T_2: Assume that there is an $f \in \mathcal{DO}(T_2) \setminus \mathcal{DO}(T_1)$. Then by definition there is an $\mathcal{L}_{BPA}$-term $t(x)$ such that

$$T_2 \vdash BigFun(a, b, t(x), X)$$

and $f \leq \big(\lambda n.\, \Phi_{\mathcal{E}}(t(n))\big) =: g$. Now $f \notin \mathcal{DO}(T_1)$ yields $g \notin \mathcal{DO}(T_1)$, hence

$$T_1 \nvdash BigFun(a, b, t(x), X).$$

12.2 Computing Dynamic Ordinals

As shown in [3] the truth complexity of the sentences $Fund(\prec, X)$ is essentially the same as $\mathcal{O}(\prec)$. Here the predicative truth complexity of $BigFun(a,b,\alpha,X)$ is closely related to $\Phi_{\mathcal{E}}(\alpha)$: The Boundedness Theorem 11.1.2 yields

$$\Phi_{\mathcal{E}}(\alpha) \leq \mathrm{ptc}(BigFun(a,b,\alpha,X(d))) \tag{12.1}$$

In Chapter 10 we gave upper bounds for $\mathrm{ptc}(BigFun(a,b,\alpha,X(d)))$. The well-ordering proofs from Chapter 9 yield lower bounds for the Dynamic Ordinals of ${}^{\mathrm{p}}\mathrm{R}^n_2$, ${}^{\mathrm{p}}\mathrm{S}^n_2$, ${}^{\mathrm{p}}\mathrm{T}^n_2$, ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n$-$\mathrm{L}^m\mathrm{Ind}$, ${}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$. Altogether this yields a sharp characterization of the Dynamic Ordinals.

12.2.1 Theorem *Let $n+1 \geq m \geq 1$, then*

$$\begin{aligned}\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind}) &= \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind}) \\ &= \mathcal{H}(\{\lambda i.2_n(p(|i|_m)) \,:\, p \text{ a polynomial}\}).\end{aligned}$$

Proof: By Lemma 7.2.5 and Theorem 7.2.6 we know

$$\begin{aligned}&{}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun(a,b,t,X) \\ &\quad\Longrightarrow\quad {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun(a,b,t,X(d)).\end{aligned}$$

Therefore we obtain

$$\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind}) \subset \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind}).$$

Let $\mathcal{F} := \{\lambda i.2_n(p(|i|_m)) \,:\, p \text{ a polynomial}\}$. By Theorem 9.3.3 we know

$${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun(a,b,\hat{2}_n(\mathrm{T}_{\mathcal{E}}(p(|x|_m))),X).$$

As $\Phi_{\mathcal{E}}(\hat{2}_n(\mathrm{T}_{\mathcal{E}}(p(|\underline{i}|_m)))) = 2_n(p(|i|_m))$ for all $i \in \omega$ this yields

$$\big(\lambda i.2_n(p(|i|_m))\big) \in \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind})$$

hence $\mathcal{H}(\mathcal{F}) \subset \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind})$.

On the other hand let t be an $\mathcal{L}_{BPA}$-term containing no other variable than x such that ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind} \vdash BigFun(a,b,t,X(d))$. We are going to convince ourself that $\lambda n.\, \Phi_{\mathcal{E}}(t(n)) \in \mathcal{H}(\mathcal{F})$. Using (12.1) it suffices to show that

$$f \;:=\; \mathrm{dptc}\left(BigFun(a, b, t, X(d)), x\right) \in \mathcal{H}(\mathcal{F})$$

with $\mathrm{dptc}(F, x_0, \ldots, x_{k-1}) \;:=\; \lambda\vec{n}.\, \mathrm{ptc}(F_{\vec{x}}(\underline{\vec{n}}))$.

As ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}$-$\mathrm{L}^m$Ind $\subset {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-$\mathrm{L}^m$Ind we obtain by Theorem 10.4.4 some term $s(x)$ containing no other variable than x and some constant $c \in \omega$ such that

$$\forall i \quad f(i) \le 2_{n+1}(c \cdot |s(i)|_{m+1}).$$

Now there is some polynomial p such that $|s(i)|_m \le p(|i|_m)$ for any $i \in \omega$, hence

$$\forall i \quad 2_{n+1}(c \cdot |s(i)|_{m+1}) \le 2_n\Big(\big(2^{|p(|i|_m)|}\big)^c\Big) \le 2_n((2 \cdot p(|i|_m) + 1)^c),$$

hence $f \in \mathcal{H}(\mathcal{F})$. This shows $\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}\text{-}\mathrm{L}^m\mathrm{Ind}) \subset \mathcal{H}(\mathcal{F})$. □

12.2.2 Corollary

$$\mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{i+1}_2) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{i+1}_2(\mathcal{X})) = \mathcal{H}(\{\lambda n.2_i(p(|n|)) \,:\, p \text{ a polynomial}\}).$$

12.2.3 Corollary

$$\begin{aligned}\mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{i+2}_2) &= \mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{i+2}_2(\mathcal{X}))\\ &= \mathcal{H}(\{\lambda n.2_{i+1}(p(||n||)) \,:\, p \text{ a polynomial}\}).\end{aligned}$$

For ${}^{\mathrm{p}}\mathrm{T}^n_2$ we can prove a sharper result:

12.2.4 Theorem

$$\begin{aligned}\mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2) &= \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X}))\\ &= \mathcal{H}(\{\lambda i.2_{n+1}(p(|i|)) \,:\, p \text{ a polynomial}\}).\end{aligned}$$

Proof: The same argument as in the proof of Theorem 12.2.1 shows

$$\mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X})) \subset \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2).$$

Let $\mathcal{F} \;:=\; \{\lambda i.2_{n+1}(p(|i|)) \,:\, p \text{ a polynomial}\}$. For each polynomial $p(x)$ containing no variable not indicated there is an $\mathcal{L}_{BPA}$-term $t'(x)$ containing no variable not indicated such that $p(|i|) \le |t'(i)|$ for all $i \in \omega$ as remarked in Chapter 2. Hence

$$2_{n+1}(p(|i|)) \le 2_n(2^{|t'(i)|}) \le 2_n(t(i))$$

for $t :\equiv \mathrm{S}_1 t'$. By Theorem 9.3.3 we know

$$\mathrm{^pT}_2^{n+1}(\mathcal{X}) \vdash BigFun(a, b, \hat{2}_n(\mathrm{T}_{\mathcal{E}}(t(x))), X).$$

As $\Phi_{\mathcal{E}}(\hat{2}_n(\mathrm{T}_{\mathcal{E}}(t(\underline{i})))) = 2_n(t(i)) \geq 2_{n+1}(p(|i|))$ for all $i \in \omega$ this yields

$$\big(\lambda i.2_{n+1}(p(|i|))\big) \in \mathcal{DO}(\mathrm{^pT}_2^{n+1}(\mathcal{X}))$$

hence $\mathcal{H}(\mathcal{F}) \subset \mathcal{DO}(\mathrm{^pT}_2^{n+1}(\mathcal{X}))$.

On the other hand let t be an $\mathcal{L}_{BPA}$-term containing no other variable than x such that $\mathrm{^pT}_2^{n+1} \vdash BigFun(a, b, t, X(d))$. In order to prove $\lambda n.\, \Phi_{\mathcal{E}}(t(n)) \in \mathcal{H}(\mathcal{F})$ it suffices to show by (12.1) that

$$f \;:=\; \mathrm{dptc}\,\big(BigFun(a, b, t, X(d)), x\big) \in \mathcal{H}(\mathcal{F}).$$

As $\mathrm{^pT}_2^{n+1} \subset \mathrm{^pT}_2^{n+1}(\mathcal{X})$ we obtain by Theorem 10.4.4 some term $s(x)$ containing no other variable than x and some constant $c \in \omega$ such that

$$\forall i \quad f(i) \leq 2_{n+1}(c \cdot |s(i)|).$$

Now there is some polynomial p such that $|s(i)| \leq p(|i|)$ for any $i \in \omega$, hence

$$\forall i \quad 2_{n+1}(c \cdot |s(i)|) \leq 2_{n+1}(c \cdot p(|i|)),$$

hence $f \in \mathcal{H}(\mathcal{F})$. This shows $\mathcal{DO}(\mathrm{^pT}_2^{n+1}) \subset \mathcal{H}(\mathcal{F})$. □

We introduce the notion "*for almost all i*" by "$\exists j\, \forall i{\geq}j$".

12.2.5 Theorem *Let $n \geq 0$ and $m \geq 1$, then*

$$\mathcal{DO}(\mathrm{^p\Sigma}^{\mathrm{b}}_{n+m}\text{-}\mathrm{L}^m\mathrm{Ind}) \subsetneq \mathcal{DO}(\mathrm{^p\Sigma}^{\mathrm{b}}_{n+m+1}\text{-}\mathrm{L}^{m+1}\mathrm{Ind}) \subsetneq \mathcal{DO}(\mathrm{^pT}_2^{n+1}).$$

Proof: By Theorems 12.2.1 and 12.2.4 we obtain "$\subset$" because for monotone polynomials p we have $p(|i|_m) \leq 2^{|p(|i|_m)|} \leq 2^{p(|i|_{m+1})}$ and

$$2_m(p(|i|_m)) < 2^i \text{ for almost all } i. \tag{12.2}$$

In order to prove $\mathcal{DO}(\mathrm{^p\Sigma}^{\mathrm{b}}_{n+m}\text{-}\mathrm{L}^m\mathrm{Ind}) \neq \mathcal{DO}(\mathrm{^p\Sigma}^{\mathrm{b}}_{n+m+1}\text{-}\mathrm{L}^{m+1}\mathrm{Ind})$ we show that

$$f \;:=\; (\lambda i.2_{n+m}((|i|_{m+1})^2)) \notin \mathcal{DO}(\mathrm{^p\Sigma}^{\mathrm{b}}_{n+m}\text{-}\mathrm{L}^m\mathrm{Ind})$$

where $f \in \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-L}^{m+1}\mathrm{Ind})$ is obvious by definition. We prove this indirectly assuming $f \in \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}\text{-L}^{m}\mathrm{Ind})$. By Theorem 12.2.1 there is some polynomial p such that

$$\forall i \quad 2_{n+m}((|i|_{m+1})^2) \leq 2_{n+m-1}(p(|i|_m)).$$

There is some k such that $p(i) \leq i^k$ for almost all i, hence for almost all i

$$2^{(|i|_{m+1})^2} \leq p(|i|_m) \leq (|i|_m)^k \leq \left(2^{|i|_{m+1}}\right)^k = 2^{k\cdot|i|_{m+1}}$$

as $|i|_m < 2^{|i|_{m+1}}$, hence

$$i^2 \leq k \cdot i$$

for almost all i because $\lambda i.|i|_{m+1}$ is surjective. A contradiction.

For $\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-L}^{m+1}\mathrm{Ind}) \neq \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2)$ we show that

$$f \;:=\; (\lambda i.2_{n+1}(|i|)) \notin \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-L}^{m+1}\mathrm{Ind})$$

where $f \in \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2)$ is obvious by definition. Towards a contradiction assume that $f \in \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-L}^{m+1}\mathrm{Ind})$. By Theorem 12.2.1 there is some polynomial p such that

$$\forall i \quad 2_{n+1}(|i|) \leq 2_{n+m}(p(|i|_{m+1})).$$

As $\lambda i.|i|$ is surjective we obtain

$$\forall i \quad 2^i \leq 2_m(p(|i|_m),$$

but this contradicts (12.2). $\square$

The same proof also yields:

12.2.6 Theorem *Let $n \geq 0$ and $m \geq 1$, then*

$$\begin{aligned}\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}(\mathcal{X})\text{-L}^{m}\mathrm{Ind}) &\subsetneq \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}(\mathcal{X})\text{-L}^{m+1}\mathrm{Ind})\\ &\subsetneq \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X})).\end{aligned}$$

12.2.7 Corollary

$$\mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+1}_2) \subsetneq \mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{n+2}_2) \subsetneq \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+2}_2) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2).$$

12.2.8 Corollary

$$\mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+1}_2(\mathcal{X})) \subsetneq \mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^{n+2}_2(\mathcal{X})) \subsetneq \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^{n+2}_2(\mathcal{X})) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^{n+1}_2(\mathcal{X})).$$

For theories T_1, T_2 let $T_1 \subseteq T_2$ iff T_1 is included in T_2, which means that for all formulas F if $T_1 \vdash F$ then $T_2 \vdash F$. Let $T_1 \subsetneq T_2$ iff T_2 is a proper extension of T_1, i.e., $T_1 \subseteq T_2$ and $T_1 \not\supseteq T_2$. As remarked at the end of the first section different Dynamic Ordinals yield a separation of the corresponding theories.

12.2.9 Corollary *Let $n \geq 0$ and $m \geq 1$, then*

$$\begin{array}{ccc} & {}^{\mathrm{p}}\mathrm{T}_2^{n+1}(\mathcal{X}) & \\ \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} \\ {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}(\mathcal{X})\text{-}\mathrm{L}^{m}\mathrm{Ind} & \subsetneq & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}(\mathcal{X})\text{-}\mathrm{L}^{m+1}\mathrm{Ind}. \end{array}$$

12.2.10 Corollary

$$\begin{array}{ccccc} {}^{\mathrm{p}}\mathrm{S}_2^{n+1}(\mathcal{X}) & \subsetneq & {}^{\mathrm{p}}\mathrm{T}_2^{n+1}(\mathcal{X}) & & \\ & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\supsetneq$} & \\ & & {}^{\mathrm{p}}\mathrm{R}_2^{n+2}(\mathcal{X}) & \subsetneq & {}^{\mathrm{p}}\mathrm{S}_2^{n+2}(\mathcal{X}). \end{array}$$

12.2.11 Corollary *Let $n \geq 0$ and $m \geq 1$, then*

$$\begin{array}{ccc} & {}^{\mathrm{p}}\mathrm{T}_2^{n+1} & \\ \rotatebox{-45}{$\supsetneq$} & & \rotatebox{45}{$\subsetneq$} \\ {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m}\text{-}\mathrm{L}^{m}\mathrm{Ind} & \subsetneq & {}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+m+1}\text{-}\mathrm{L}^{m+1}\mathrm{Ind}. \end{array}$$

12.2.12 Corollary

$$\begin{array}{ccccc} {}^{\mathrm{p}}\mathrm{S}_2^{n+1} & \subsetneq & {}^{\mathrm{p}}\mathrm{T}_2^{n+1} & & \\ & \rotatebox{45}{$\supsetneq$} & & \rotatebox{-45}{$\supsetneq$} & \\ & & {}^{\mathrm{p}}\mathrm{R}_2^{n+2} & \subsetneq & {}^{\mathrm{p}}\mathrm{S}_2^{n+2}. \end{array}$$

12.3 DOA in theories of BA

As $\mathcal{L}_{BA}$ does not contain impredicative variables we have to modify the definition of the Dynamic Ordinals for theories of bounded arithmetic.

12.3.1 Definition *Let T be a theory formulated in $\mathcal{L}_{BA}(\mathcal{X})$. We define the* Dynamic Ordinal of T *by*

$$\begin{aligned}\mathcal{DO}(T) := \mathcal{H}\big(\{\lambda n.\Phi_{\mathcal{E}}(t(n)) | t(x) \text{ is an } \mathcal{L}_{BA}\text{-term with } \mathsf{FV}(t) \subset \{x\} \\ \text{and there are } \mathcal{L}_{BA}\text{-terms } s_1(x), s_2(x) \text{ with} \\ \mathsf{FV}(s_1, s_2) \subset \{x\} \text{ such that } \mathbb{N} \vDash \forall x\, Big(s_1, s_2, t) \\ \text{and } T \vdash \forall x\, BigFun(s_1, s_2, t, X)\}\big).\end{aligned}$$

In Section 2 of this chapter we have seen that the functions $2_n(p(|x|_{n+1}))$ resp. $2^{p(|x|)}$ yield a good measurement of the theories ${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-$\mathrm{L}^{n+1}$Ind resp. ${}^{\mathrm{p}}\mathrm{T}^1_2(\mathcal{X})$ and ${}^{\mathrm{p}}\mathrm{S}^2_2(\mathcal{X})$ in the sense described at the beginning of this chapter. All these functions can be majorized by an $\mathcal{L}_{BA}$-term t with $\mathsf{FV}(t) \subset \{x\}$. By Lemma 9.1.2 there are $\mathcal{L}_{BA}$-terms s_1, s_2 with $\mathsf{FV}(s_1, s_2) \subset \{x\}$ such that $\mathbb{N} \vDash \forall x\, Big(s_1, s_2, \mathrm{T}_{\mathcal{E}}(t))$. Replacing s_1 and s_2 for a resp. b in the well-ordering proof Theorem 9.3.3 and using the conservativity results from Theorem 8.4.3 we obtain

$$\begin{aligned}\mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^{n+1}\mathrm{Ind}) \subset \mathcal{DO}(\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^{n+1}\mathrm{Ind}) \\ \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^1_2(\mathcal{X})) \subset \mathcal{DO}(\mathrm{T}^1_2(\mathcal{X})) \\ \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^2_2(\mathcal{X})) \subset \mathcal{DO}(\mathrm{S}^2_2(\mathcal{X})).\end{aligned} \tag{12.3}$$

For the other inclusions assume $T \vdash BigFun(s_1, s_2, t, X)$ where T is one of $\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})$-$\mathrm{L}^{n+1}$Ind, $\mathrm{T}^1_2(\mathcal{X})$, $\mathrm{S}^2_2(\mathcal{X})$ and s_1, s_2, t are $\mathcal{L}_{BA}$-terms with $\mathsf{FV}(s_1, s_2, t) \subset \{x\}$ such that $\mathbb{N} \vDash \forall x\, Big(s_1, s_2, t)$. An inspection of the proof of the Predicative Boundedness Theorem 11.1.1 yields

$${}^{\mathrm{p}}\vert\!\frac{m}{1,1}\, BigFun(\underline{a}, \underline{b}, \underline{\alpha}, X) \implies \Phi_{\mathcal{E}}(\underline{\alpha}) \le m$$

for numerals $\underline{a}, \underline{b}, \underline{\alpha}$ satisfying $\mathbb{N} \vDash Big(\underline{a}, \underline{b}, \underline{\alpha})$. Thus, in order to majorize $\lambda n.\, \Phi_{\mathcal{E}}(t_x(n))$ it suffices to dominate

$$\mathrm{dptc}(BigFun(s_1, s_2, t, X), x).$$

But as the predicative version of T is an extension of T the same estimations from the proofs in Section 2 yield the other inclusions of (12.3). Thus, we have shown

12.3.2 Theorem

$$
\begin{aligned}
\mathcal{DO}(\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^{n+1}\mathrm{Ind}) &= \mathcal{DO}({}^{\mathrm{p}}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^{n+1}\mathrm{Ind}) \\
&= \mathcal{H}(\{\lambda i.2_n(p(|i|_{n+1}))\ :\ p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{S}^1_2(\mathcal{X})) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^1_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.p(|i|)\ :\ p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{sR}^2_2(\mathcal{X})) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{R}^2_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.2^{p(||i||)}\ :\ p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{T}^1_2(\mathcal{X})) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{T}^1_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.2^{p(|i|)}\ :\ p \text{ a polynomial}\}) \\
\mathcal{DO}(\mathrm{S}^2_2(\mathcal{X})) = \mathcal{DO}({}^{\mathrm{p}}\mathrm{S}^2_2(\mathcal{X})) &= \mathcal{H}(\{\lambda i.2^{p(|i|)}\ :\ p \text{ a polynomial}\}).
\end{aligned}
$$

12.3.3 Corollary *Let $m \geq 1$, then*

$$
\begin{array}{ccccc}
 & & \mathrm{T}^1_2(\mathcal{X}) & & \\
 & \not\subseteq & & \not\supseteq & \\
\mathrm{s}\Sigma^{\mathrm{b}}_m(\mathcal{X})\text{-}\mathrm{L}^m\mathrm{Ind} & & \subsetneqq & & \mathrm{s}\Sigma^{\mathrm{b}}_{m+1}(\mathcal{X})\text{-}\mathrm{L}^{m+1}\mathrm{Ind}.
\end{array}
$$

12.3.4 Corollary

$$
\begin{array}{ccccc}
\mathrm{S}^1_2(\mathcal{X}) & \subsetneqq & \mathrm{T}^1_2(\mathcal{X}) & & \\
 & \subsetneqq & & \not\supseteq & \\
 & & \mathrm{sR}^2_2(\mathcal{X}) & \subsetneqq & \mathrm{S}^2_2(\mathcal{X}).
\end{array}
$$

12.3.5 Remark *Observe that the usually considered minimization axioms*

$$
Min(F, y, x) \equiv \exists y{\leq}x\, F \to \exists y{\leq}x\, [F \wedge \forall z{<}y\, (\neg F_y(z))]
$$

can also serve as a separation formula because $Min((y \in X), y, x)$ is similar to $Fund(x, X)$. An inspection of the proofs yields

$$
\mathrm{T}^1_2(\mathcal{X}) \vdash Min((y \in X), y, x)
$$

but

$$
\mathrm{s}\Sigma^{\mathrm{b}}_{n+1}(\mathcal{X})\text{-}\mathrm{L}^{n+1}\mathrm{Ind} \nvdash Min((y \in X), y, x),
$$

hence

$$
\mathrm{S}^1_2(\mathcal{X}) \nvdash Min((y \in X), y, x)
$$

and

$$
\mathrm{sR}^2_2(\mathcal{X}) \nvdash Min((y \in X), y, x).
$$

Appendix A

Standard interpretations

Before we give an axiomatization of the predicates from $\mathcal{P}^i$ and functions from $\mathcal{F}^i$ we will fix the standard interpretation $\underline{\mathrm{P}}^{\mathbb{N}} \subset \omega^{\mathrm{ar}(\underline{\mathrm{P}})}$ for $P \in \mathcal{P}^i$ and ${\mathcal{G}_f}^{\mathbb{N}} \subset \omega^{\mathrm{ar}(f)+1}$ for $f \in \mathcal{F}^i$ which we have in mind. Remember that by definition $\underline{\mathrm{P}^{\mathrm{c}}}^{\mathbb{N}} = \omega^{\mathrm{ar}(\underline{\mathrm{P}})} \setminus \underline{\mathrm{P}}^{\mathbb{N}}$ and $\mathcal{G}_f^{c\,\mathbb{N}} = \omega^{\mathrm{ar}(f)+1} \setminus {\mathcal{G}_f}^{\mathbb{N}}$. For graphs $S \subset \omega^{k+1}$ which potentially define a partial function we define the totalization by

$$\mathrm{Tot}_0(S) \;:=\; S \cup \{(\vec{x}, 0) \in \omega^{k+1} \;:\; \neg(\exists y{\in}\omega)((\vec{x}, y) \in S)\}.$$

- $\underline{\leq}^{\mathbb{N}} = \leq$, $\underline{\mathrm{Bit}}^{\mathbb{N}} = \mathrm{Bit}$, $\underline{\mathtt{Seq}}^{\mathbb{N}} = \mathtt{Seq}$, $\underline{\mathcal{E}}^{\mathbb{N}} = \mathcal{E}$, $\underline{\prec}^{\mathbb{N}} = \prec \cap (\mathcal{E} \times \mathcal{E})$,
- for $f \in \{\mathrm{S}, \mathrm{S}_0, \mathrm{S}_1\}$ we set ${\mathcal{G}_f}^{\mathbb{N}} = \{(x, y) \;:\; f(x) = y\}$.
- ${\mathcal{G}_*}^{\mathbb{N}} = \mathrm{Tot}_0(\{(s, x, u) \;:\; s \in \mathtt{Seq} \;\&\; s * x = u\})$.
- ${\mathcal{G}_{**}}^{\mathbb{N}} = \mathrm{Tot}_0(\{(s, t, u) \;:\; s, t \in \mathtt{Seq} \;\&\; s ** t = u\})$.
- for $f \in \{\mathrm{first}, \mathrm{last}, \mathrm{trunc}_\mathrm{l}, \mathrm{trunc}_\mathrm{r}, \mathrm{lh}\}$ we set
  ${\mathcal{G}_f}^{\mathbb{N}} = \mathrm{Tot}_0(\{(s, u) \;:\; s \in \mathtt{Seq} \;\&\; f(s) = u\})$.
- ${\mathcal{G}_\beta}^{\mathbb{N}} = \mathrm{Tot}_0(\{(i, s, u) \;:\; s \in \mathtt{Seq} \;\&\; \beta(i, s) = u\})$.
- ${\mathcal{G}_{\hat{0}}}^{\mathbb{N}} = \{\hat{0}\}$, ${\mathcal{G}_{\hat{1}}}^{\mathbb{N}} = \{\hat{1}\}$,
- for $f \in \{\hat{2}^{\cdot}, \widehat{2\cdot}\,.\}$ we set
  ${\mathcal{G}_f}^{\mathbb{N}} = \mathrm{Tot}_0(\{(\alpha, u) \;:\; \alpha \in \mathcal{E} \;\&\; f(\alpha) = u\})$.
- for $f \in \{.\check{+}\check{2}^{\cdot}, \hat{+}\}$ we set
  ${\mathcal{G}_f}^{\mathbb{N}} = \mathrm{Tot}_0(\{(\alpha, \beta, u) \;:\; \alpha, \beta \in \mathcal{E} \;\&\; f(\alpha, \beta) = u\})$.
- ${\mathcal{G}_{\mathrm{T}_\mathcal{E}}}^{\mathbb{N}} = \{(x, u) \;:\; \mathrm{T}_\mathcal{E}(x) = u\}$.

Appendix B

${}^{\mathrm{p}}$BASIC

We obtain the axiom needed to define ${}^{\mathrm{p}}$BASIC by applying the transformation $el_{\mathcal{F}^i}$ described in Chapter 9 to the axioms listed below. E.g. the axiom **Less.5** is transformed in the following way:

$$(a < \mathrm{S}_1\, a)^{el_{\mathcal{F}^i}} \quad :- \quad \mathcal{G}^c_{\mathrm{S}_1}(a, b) \vee a < b.$$

Axioms for $<$

Less.1 $a < b < c \rightarrow a < c$

Less.2 $a < b \vee a = b \vee b < a$

Less.3 $a \not< a$

Less.4 $a \neq 0 \rightarrow a < \mathrm{S}_0\, a$

Less.5 $a < \mathrm{S}_1\, a$

Successor axioms

Suc.1 $\mathrm{S}\, 0 = \mathrm{S}_1\, 0$

Suc.2 $a \neq 0 \rightarrow \mathrm{S}(\mathrm{S}_0\, a) = \mathrm{S}_1\, a$

Suc.3 $\mathrm{S}(\mathrm{S}_1\, a) = \mathrm{S}_0(\mathrm{S}\, a)$

Bit.1 $\mathrm{Bit}(n, 0) = 0$

Bit.2 $\mathrm{Bit}(0, \mathrm{S}_i\, a) = i$ for $i \in \{0, 1\}$

Bit.3 $\mathrm{Bit}(\mathrm{S}\, n, \mathrm{S}_i\, a) = \mathrm{Bit}(n, a)$ for $i \in \{0, 1\}$

Sequence axioms

Seq.1 $\langle\rangle = 0$

Seq.2 $0 \in \mathtt{Seq}$

Seq.3 $s \in \mathtt{Seq} \leftrightarrow \mathrm{S_i}(\mathrm{S_1}(s)) \in \mathtt{Seq}$ for $i \in \{0,1\}$

Seq.4 $s \in \mathtt{Seq} \leftrightarrow \mathrm{S_i}(\mathrm{S_1}(\mathrm{S_0}(\mathrm{S_0}(s)))) \in \mathtt{Seq}$ for $i \in \{0,1\}$

Seq.5 $s \in \mathtt{Seq} \leftrightarrow \mathrm{S_i}(\mathrm{S_1}(\mathrm{S_1}(\mathrm{S_0}(s)))) \notin \mathtt{Seq}$ for $i \in \{0,1\}$

Seq.6 $s \in \mathtt{Seq} \rightarrow s * 0 = \mathrm{S_0}(\mathrm{S_1}(\mathrm{S_0}(\mathrm{S_0}(s))))$

Seq.7 $s \in \mathtt{Seq} \rightarrow s * (\mathrm{S_i}\, a) = \mathrm{S_i}(\mathrm{S_1}(s * a))$ for $i \in \{0,1\}$

Seq.8 $\langle a\rangle = \langle\rangle * a$

Seq.9 $s \in \mathtt{Seq} \rightarrow a < s * a \wedge s < s * a$

Seq.10 $s \in \mathtt{Seq} \rightarrow s ** \langle\rangle = s$

Seq.11 $s, t \in \mathtt{Seq} \rightarrow s ** (t * a) = (s ** t) * a$

Seq.12 $\mathrm{first}(\langle a\rangle) = a$

Seq.13 $s \in \mathtt{Seq} \wedge s \neq 0 \rightarrow \mathrm{first}(s * a) = \mathrm{first}(s)$

Seq.14 $s \in \mathtt{Seq} \rightarrow \mathrm{last}(s * a) = a$

Seq.15 $\mathrm{trunc_l}(\langle a\rangle) = \langle\rangle$

Seq.16 $s \in \mathtt{Seq} \wedge s \neq 0 \rightarrow \mathrm{trunc_l}(s * a) = \mathrm{trunc_l}(s) * a$

Seq.17 $s \in \mathtt{Seq} \rightarrow \mathrm{trunc_r}(s * a) = s$

Seq.18 $\beta(i, \langle\rangle) = 0$

Seq.19 $s \in \mathtt{Seq} \rightarrow \beta(0, s * a) = \mathrm{S}(\beta(0, s))$

Seq.20 $s \in \mathtt{Seq} \rightarrow \beta(\mathrm{S}\,0, s) = \mathrm{first}(s)$

Seq.21 $s \in \mathtt{Seq} \wedge i > 0 \rightarrow \beta(\mathrm{S}\,i, s) = \beta(i, \mathrm{trunc_l}(s))$

Seq.22 $s \in \mathtt{Seq} \rightarrow \mathrm{lh}(s) = \beta(0, s)$

Axioms for exponential notations

Exp.1 $\alpha, \beta, \gamma \in \mathcal{E} \wedge \alpha \prec \beta \prec \gamma \to \alpha \prec \gamma$

Exp.2 $\alpha, \beta \in \mathcal{E} \to \alpha \prec \beta \vee \alpha = \beta \vee \beta \prec \alpha$

Exp.3 $\alpha \in \mathcal{E} \to \neg \alpha \prec \alpha$

Exp.4 $\hat{0} = \langle \rangle$

Exp.5 $\hat{2}^{a} = \langle a \rangle$

Exp.6 $s \in \mathtt{Seq} \to s \check{+} \check{2}^{b} = s * b$

Exp.7 $\hat{1} = \hat{2}^{\hat{0}}$

Exp.8 $\alpha \in \mathcal{E} \to \alpha \in \mathtt{Seq}$

Exp.9 $\hat{0} \in \mathcal{E}$

Exp.10 $\alpha \in \mathcal{E} \to \hat{2}^{\alpha} \in \mathcal{E}$

Exp.11 $\alpha, \beta \in \mathcal{E} \wedge \beta \prec \mathrm{last}(\alpha) \to \alpha \check{+} \check{2}^{\beta} \in \mathcal{E}$

Exp.12 $\alpha \in \mathcal{E} \wedge \alpha \neq \hat{0} \to \alpha = \mathrm{trunc_r}(\alpha) \check{+} \check{2}^{\mathrm{last}(\alpha)} \wedge$
$\mathrm{trunc_r}(\alpha), \mathrm{last}(\alpha) \in \mathcal{E} \wedge$
$[\mathrm{trunc_r}(\alpha) = \hat{0} \vee \mathrm{last}(\alpha) \prec \mathrm{last}(\mathrm{trunc_r}(\alpha))]$

Exp.13 $\alpha \prec \beta \leftrightarrow \alpha, \beta \in \mathcal{E} \wedge$
$[(\alpha = \hat{0} \wedge \beta \neq \hat{0}) \vee (\mathrm{first}(\alpha) \prec \mathrm{first}(\beta)) \vee$
$(\mathrm{first}(\alpha) = \mathrm{first}(\beta) \wedge \mathrm{trunc_l}(\alpha) \prec \mathrm{trunc_l}(\beta))]$

Exp.14 $\alpha, \beta \in \mathcal{E} \to \alpha \hat{+} \beta = \beta \hat{+} \alpha$

Exp.15 $\alpha, \beta, \gamma \in \mathcal{E} \to (\alpha \hat{+} \beta) \hat{+} \gamma = \alpha \hat{+} (\beta \hat{+} \gamma)$

Exp.16 $\gamma \in \mathcal{E} \to \hat{2}^{\gamma} \hat{+} \hat{2}^{\gamma} = \hat{2}^{\gamma \hat{+} \hat{1}}$

Exp.17 $\alpha \check{+} \check{2}^{\gamma} \in \mathcal{E} \to \alpha \check{+} \check{2}^{\gamma} = \alpha \hat{+} \hat{2}^{\gamma}$

Exp.18 $\alpha, \beta \in \mathcal{E} \wedge \alpha \prec \beta \to \hat{2}^{\alpha} \prec \hat{2}^{\beta}$

Exp.19 $\alpha, \beta, \gamma, \delta \in \mathcal{E} \wedge \alpha \prec \beta \wedge \gamma \preceq \delta \to \alpha \hat{+} \gamma \prec \beta \hat{+} \delta$

Exp.20 $\alpha \in \mathcal{E} \to \alpha \prec \hat{2}^{\alpha}$

Exp.21 $\alpha, \beta \in \mathcal{E} \to (\alpha \prec \beta \leftrightarrow \alpha \mathbin{\hat{+}} \hat{1} \preceq \beta)$

Exp.22 $\widehat{2\cdot}\, \hat{0} = \hat{0}$

Exp.23 $(\alpha \check{+} \check{2}^{\beta}) \in \mathcal{E} \to \widehat{2\cdot}(\alpha \check{+} \check{2}^{\beta}) = (\widehat{2\cdot}\, \alpha) \check{+} \check{2}^{(\beta \mathbin{\hat{+}} \hat{1})}$

Exp.24 $\mathrm{T}_{\mathcal{E}}(0) = \hat{0}$

Exp.25 $\mathrm{T}_{\mathcal{E}}(\mathrm{S}_0\, a) = \widehat{2\cdot}(\mathrm{T}_{\mathcal{E}}(a))$

Exp.26 $\mathrm{T}_{\mathcal{E}}(\mathrm{S}_1\, a) = \widehat{2\cdot}(\mathrm{T}_{\mathcal{E}}(a)) \mathbin{\hat{+}} \hat{1}$

Exp.27 $\mathrm{T}_{\mathcal{E}}(\mathrm{S}\, a) = \mathrm{T}_{\mathcal{E}}(a) \mathbin{\hat{+}} \hat{1}$

Exp.28 $a < b \leftrightarrow \mathrm{T}_{\mathcal{E}}(a) \prec \mathrm{T}_{\mathcal{E}}(b)$

Appendix C

Proving indiscernibility

We give a detailed proof of the Main Lemma 11.2.3. For atomic $\mathcal{L}^p_\omega$-formulas F other than $\underline{\mathrm{Bit}}(\cdot,d)$ or $\underline{\mathrm{Bit}^{\mathrm{c}}}(\cdot,d)$, satisfying $\mathsf{FV}(F) \subset \{d\}$ and $F \in \textsc{gb}(l)$, we show

$$\forall d \in \textsc{Isc}_l \ \mathbb{N} \models F \quad \textit{or} \quad \forall d \in \textsc{Isc}_l \ \mathbb{N} \not\models F. \tag{C.1}$$

Clearly the assertion (C.1) only has to be proved either for a relation P or its complement $\overline{P}$. Let u, v be some ground terms with $u, v \leq l$, $v > 0$ and let s, t be some $\textsc{gb}(l)$-terms with $\mathsf{FV}(s,t) \subset \{d\}$. If, in the sequel, we speak of "always" we mean "for all l-indiscernibles". Let $d \in \textsc{Isc}_l$ and observe

$$u, v \leq l < 2^{2^l} < d \quad \textit{and} \quad s, t \leq d.$$

- $d \underline{\leq} u$ is always false and $s \underline{\leq} d$ is always true.
- $\underline{\mathrm{Bit}}(d,u)$ is always false, for $|u| \leq u < d$, thus the d-th bit in the binary expansion of u is always 0.

Now we check the cases for $\mathcal{G}_f$ for $f \in \{\mathrm{S}, \mathrm{S}_0, \mathrm{S}_1\}$:

- $\mathcal{G}_f(u,d)$ is always false, because $f(u) \leq 2{\cdot}u{+}1 \leq 2{\cdot}l{+}1 < 2^{2^l} < d$.
- $\mathcal{G}_f(d,s)$ is always false, because $f(d) > d \geq s$.

For the next cases we repeat the essential observation, that no l-indiscernible is a sequence-number, because the highest bits of an indiscernible always have the form $10010\ldots$ which cannot be the highest bits of a sequence-number: sequence-numbers are build up from $00, 10, 11$, hence 1001 and 010010 are impossible beginnings.

- $d \in \underline{\texttt{Seq}}$ is always false, see above.
- $\mathcal{G}_{*}(d,s,0)$, $\mathcal{G}_{**}(d,s,0)$, $\mathcal{G}_{**}(s,d,0)$ are always true, because $d \notin \texttt{Seq}$.
- $\mathcal{G}_{*}(s,t,d)$, $\mathcal{G}_{*}(d,s,v)$ are always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.
- $\mathcal{G}_{*}(u,d,v)$ is always false, because if $u \in \texttt{Seq}$ then $u * d > d > v$ and if $u \notin \texttt{Seq}$ then $v > 0$.
- $\mathcal{G}_{**}(s,t,d), \mathcal{G}_{**}(s,d,v), \mathcal{G}_{**}(d,u,v)$ are always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.

We check the cases for $f \in \{\mathrm{first}, \mathrm{last}, \mathrm{trunc}_{\mathrm{l}}, \mathrm{trunc}_{\mathrm{r}}, \mathrm{lh}\}$:

- $\mathcal{G}_f(d,0)$ is always true, because $d \notin \texttt{Seq}$.
- $\mathcal{G}_f(d,d)$, $\mathcal{G}_f(d,v)$ is always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.
- $\mathcal{G}_f(u,d)$ is always false, because if $u \in \texttt{Seq}$ then $f(u) \leq u < d$ and if $u \notin \texttt{Seq}$ then $d > 0$.
- $\mathcal{G}_{\beta}(s,d,0)$ is always true, because $d \notin \texttt{Seq}$.
- $\mathcal{G}_{\beta}(s,d,d)$, $\mathcal{G}_{\beta}(s,d,v)$ is always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.
- $\mathcal{G}_{\beta}(s,u,d)$ is always false, because if $u \in \texttt{Seq}$ then $\beta(s,u) \leq u < d$ and if $u \notin \texttt{Seq}$ then $d > 0$.
- $\mathcal{G}_{\beta}(d,u,v)$ is always false, because $v > 0$ and if $u \in \texttt{Seq}$ then $d > \mathrm{lh}(u)$.
- $d \in \underline{\mathcal{E}}$ is always false, because $d \notin \texttt{Seq}$ and $\mathcal{E} \subset \texttt{Seq}$.
- $s \underline{\preceq} d, d \underline{\preceq} u$ are always false, because $d \notin \texttt{Seq}$.

For $f \in \{.\check{+}\check{2}\cdot, \hat{+}\}$ we observe

- $\mathcal{G}_f(d,s,0)$, $\mathcal{G}_f(s,d,0)$ are always true, because $d \notin \texttt{Seq}$.
- $\mathcal{G}_f(s,t,d)$, $\mathcal{G}_f(s,d,v)$, $\mathcal{G}_f(d,u,v)$ are always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.

For $f \in \{\hat{2}, \widehat{2\cdot}\}$ we observe

- $\mathcal{G}_f(d,0)$ is always true, because $d \notin \texttt{Seq}$.
- $\mathcal{G}_f(s,d), \mathcal{G}_f(d,v)$ are always false, because $d \notin \texttt{Seq}$ and $v,d > 0$.

- $\mathcal{G}_{\mathrm{T}_{\mathcal{E}}}(s,d)$ is always false, because $d \notin \mathtt{Seq}$.

- $\mathcal{G}_{\mathrm{T}_{\mathcal{E}}}(d,u)$ is always false, because: if $u \notin \mathcal{E}$ this is clear, and if $u \in \mathcal{E}$, then $d > h(l) \geq \max\{\Phi_{\mathcal{E}}(\alpha) \, : \, \alpha \in \mathcal{E} \cap (l+1)\} \geq \Phi_{\mathcal{E}}(u)$, hence $\mathrm{T}_{\mathcal{E}}(d) \neq u$.

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Index

Notations

PH, **PSPACE**, **EXPTIME**, 1
P, **NP**, 1
$\mathbf{P}^A$, $\mathbf{NP}^A$, $\mathbf{PH}^A$, 2
$\Sigma^{\mathrm{b}}_n(\mathcal{X})$, 2
$\mathrm{s}\Sigma^{\mathrm{b}}_n(\mathcal{X})$, 2
$\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$, 2
$\mathrm{sR}^n_2(\mathcal{X})$, $\mathrm{R}^n_2(\mathcal{X})$, 2
$\mathrm{S}^n_2(\mathcal{X})$, $\mathrm{T}^n_2(\mathcal{X})$, 2
$\mathcal{O}(S)$, 3
$Fund(\prec, X)$, 4
$\mathrm{I}\Sigma^0_1$, $\Sigma^0_1(\mathcal{X})$, 4
$f \le g$, $\mathcal{H}(\mathcal{F})$, $\mathcal{DO}(T)$, 5
$Fund(a, \alpha, X)$, 8
${}^{\mathrm{p}}\Sigma^{\mathrm{b}}_n(\mathcal{X})$-$\mathrm{L}^m\mathrm{Ind}$, 8
${}^{\mathrm{p}}\mathrm{R}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{S}^n_2(\mathcal{X})$, ${}^{\mathrm{p}}\mathrm{T}^n_2(\mathcal{X})$, 8
I_p, I, 8
$Big(a, b, \alpha)$, 9
$T_1 \subseteq T_2$, $T_1 \subsetneq T_2$, 10
$\omega, \mathfrak{P}(\omega)$, 13
Ω, 13
$\omega_n(\alpha)$, $2_n(m)$, 13
$(i_k \ldots i_0)_2$, 13
$S_\varphi(t)$, 13
$A(\cdot)$, 13
S, $+$, $\cdot$, 13
S_0, S_1, 14
$|n|$, $|\vec{n}|$, 14
$\lfloor \frac{1}{2} n \rfloor$, 14
$m \# n$, 14
$m \dot{-} n$, 14
$\mathrm{MSP}(m, n)$, $\mathrm{LSP}(m, n)$, 14
$<$, $\le$, 14
$\mathrm{Bit}(m, n)$, 14
$\mathcal{L}_{Z_1}$, 17
$x_0, x_1, \ldots, X_0, X_1, \ldots$, 17
$\underline{n}$, $\underline{f}$, 17
$\neg$, 17
$F \to G$, $F \leftrightarrow G$, 17
$t^{\mathbb{N}}$, 18
$\forall x{\le}t\, A$, $\exists x{\le}t\, A$, 18
$\mathbb{N} \models F_{\vec{Y}}[\vec{M}]$, $\mathbb{N} \models F$, 18
Σ^0_i, Π^0_i, Δ^0_0, 18
$\mathrm{BASIC}_{\mathrm{Z}_1}$, 18
$(\Phi$-IND$)$, 19
Z_1, $\mathrm{I}\Sigma^0_n$, 19
$T \vdash F$, $\vdash F$, 19
field$(\prec)$, 19
$|n|_\prec$, $|| \prec ||$, 19
$Prog(\prec, X)$, $Fund(\prec, X)$, 19
$\mathcal{O}(T)$, 19
ε_0, 19
$=_{\mathrm{CNF}}$, $\ulcorner \alpha \urcorner$, 20
$\mathcal{D}$, $\prec$, $\hat{+}$, $\hat{\cdot}$, $\hat{\omega}^{\cdot}$, 20
$\forall \alpha\, A(\alpha)$, $\exists \alpha\, A(\alpha)$, 21
$\forall \beta{\le}\alpha\, A(\beta)$, $\exists \beta{\le}\alpha\, A(\beta)$, 21
$\forall \beta{<}\alpha\, A(\beta)$, $\exists \beta{<}\alpha\, A(\beta)$, 21
$\alpha \subset X$, 21
$\beta \preceq \alpha$, 21

$Prog(\alpha, X)$, $Fund(\alpha, X)$, 21
$F(A(\cdot))$, 21
$Jp(\alpha, X)$, 22
$Jp_k(\alpha, X)$, 24
$\mathcal{L}_\infty$, $\bigwedge$, $\bigvee$, 25
$\bigwedge_{i\in I} A_i$, $\bigvee_{i\in I} A_i$, 25
$\mathbb{N} \vDash F_{\vec{Y}}[\vec{N}]$, 25
$\neg$, 26
$\mathcal{L}_\Omega$, $\mathcal{L}_\omega$, 26
F^*, 26
$\mathrm{rk}(F)$, 26
$\frac{\alpha}{\rho}\!\!\vdash \Delta$, 27
$\mathrm{tc}(F)$, 27
$\mathrm{I}\Phi \vdash \Delta$, 27
$\mathrm{IND_n} \frac{\alpha}{\rho}\!\!\vdash \Delta$, 28
$\mathrm{S}(F)$, F^f, 34
$\mathrm{lh}(F)$, 35
$\mathrm{N}_X(\Delta)$, 37
$\mathrm{A}_\prec(S)$, $\mathrm{A}^\alpha_\prec(S)$, 37
$\mathrm{en}_\mathcal{O}$, $\overline{\mathrm{en}}_\mathcal{O}$, $C^\prec$, 38
$\mathrm{R}^\alpha_\prec(C)$, 38
$*$, $\langle\ldots\rangle$, $\mathtt{Seq}$, 46
$**$, β, lh, SqBd, 46
$\mathrm{trunc_r}$, $\mathrm{trunc_l}$, first, last, 46
$\hat{0}$, $\check{2}^{\alpha_1}\check{+}\ldots\check{+}\check{2}^{\alpha_k}$, $\hat{1}$, 47
$\mathcal{E}$, $\prec$, $\mathrm{T}_\mathcal{E}$, $\Phi_\mathcal{E}$, $\hat{+}$, $\hat{2}$, 47
$\mathcal{E}$, $\prec$, 51
$\hat{\mathrm{S}}$, 52
$\hat{+}$, 54
$\mathrm{T}_\mathcal{E}$, 55
$\hat{\mathrm{P}}$, 56
BPA, 57
$f(\vec{x};\vec{a})$, 57
I_p, $\mathcal{F}^p$, I, $\mathcal{F}^i$, 58
$\mathcal{L}_{BPA}$, 58
$x_0, x_1, \ldots, y, z, \ldots$, 58
$a_0, a_1, \ldots, b, c, \ldots$, 58
$\underline{\mathrm{f}}$, 58
$\mathcal{G}_f$,$\mathcal{G}^c_f$, 59
$\mathcal{P}^i$, $\underline{\mathrm{P}}$, $\underline{\mathrm{P}^c}$, 59
$\mathcal{L}_{BPA}(\mathcal{X})$, 59
$X_0, X_1, \ldots, Y, Z, \ldots$, 59
φ, 59
$\neg$, 59
$\mathsf{FV}(F)$, $\mathsf{nFV}(F)$, $\mathsf{sFV}(F)$, 60
$t^\mathcal{S}[\Phi]$, 60
$\mathbb{N}$, $\mathbb{N} \vDash F_\varphi[n]$, 60
$\forall\varphi{\leq}u\,A$, $\exists\varphi{\leq}u\,A$, 60
$\forall\varphi{<}u\,A$, $\exists\varphi{<}u\,A$, 60
PBF, PBF($\mathcal{X}$), 60
${}^\mathrm{p}\Sigma^\mathrm{b}_n$, ${}^\mathrm{p}\Pi^\mathrm{b}_n$, ${}^\mathrm{p}\Delta^\mathrm{b}_0$, ${}^\mathrm{p}\Sigma^\mathrm{b}_\infty$, 61
${}^\mathrm{p}\Sigma^\mathrm{b}_n(\mathcal{X})$, ${}^\mathrm{p}\Pi^\mathrm{b}_n(\mathcal{X})$, 61
${}^\mathrm{p}\Delta^\mathrm{b}_0(\mathcal{X})$, ${}^\mathrm{p}\Sigma^\mathrm{b}_\infty(\mathcal{X})$, 61
${}^\mathrm{p}$BASIC, 61
$Ind(F, x)$, $PInd(F, x)$, 62
$|n|_m$, 62
Ψ-Ind, Ψ-LInd, Ψ-PInd, 62
Ψ-LLInd, Ψ-PLInd, 62
Ψ-L^mInd, Ψ-PL^mInd, 62
${}^\mathrm{p}\mathrm{R}^n_2$, ${}^\mathrm{p}\mathrm{S}^n_2$, ${}^\mathrm{p}\mathrm{T}^n_2$, 62
${}^\mathrm{p}\mathrm{R}^n_2(\mathcal{X})$, ${}^\mathrm{p}\mathrm{S}^n_2(\mathcal{X})$, ${}^\mathrm{p}\mathrm{T}^n_2(\mathcal{X})$, 62
T-Ind, 63
$T \vdash \Delta$, 63
$F_X(\{a : G(a)\})$, $F_X(G(.))$, 64
BA, 67
$T_1 \subset T_2$, $T_1 = T_2$, 67
$\mathcal{L}_{BA}$, $\mathcal{L}_{BA}(\mathcal{X})$, 67
Σ^b_n, Π^b_n, Δ^b_0, 67
$\Sigma^\mathrm{b}_n(\mathcal{X})$, $\Pi^\mathrm{b}_n(\mathcal{X})$, $\Delta^\mathrm{b}_0(\mathcal{X})$, 68
$\mathrm{s}\Sigma^\mathrm{b}_n$, $\mathrm{s}\Pi^\mathrm{b}_n$, $\mathrm{s}\Delta^\mathrm{b}_0$, 68
$\mathrm{s}\Sigma^\mathrm{b}_n(\mathcal{X})$, $\mathrm{s}\Pi^\mathrm{b}_n(\mathcal{X})$, $\mathrm{s}\Delta^\mathrm{b}_0(\mathcal{X})$, 68
BASIC, 68
sR^n_2, R^n_2, S^n_2, T^n_2, 69
$BB(F, y_0, y_1, x_0, x_1)$, BB Ψ, 69

$\forall\beta\, A(\beta)$, $\exists\beta\, A(\beta)$, 75
$\forall\beta \prec \alpha\, A(\beta)$, $\exists\beta \prec \alpha\, A(\beta)$, 75
$(\forall u\, A)^t$, $(\exists u\, A)^t$, 75
$\alpha \sqsubset X$, 76
$Prog(a, \alpha, X)$, 76
$Fund(a, \alpha, X)$, 76
$T \vdash F \in \Pi$, 76
$Big(a, b, \alpha)$, 76
$\mathcal{L}_{BPA}(\mathcal{X}, \mathcal{F}^i)$, $F^{el_{\mathcal{F}^i}}$, 78
$Jp(a, \alpha, X)$, 79
$Jp_k(a, \alpha, X)$, 82
$BigFun(a, b, \alpha, X)$, 85
$X(d)$, 86
$\mathcal{L}^p_\omega$, 87
F^{*_p}, 88
Γ^{*_p}, 88
$\mathrm{prk}(F)$, 88
$\sigma[t]$, 88
$\mathrm{plh}(F)$, 89
$\textsc{bsf}^{\mathrm{p}}$, 90
$\mathrm{ptc}(F)$, 91
$\mathrm{dptc}(F, \vec{x})$, 92
$\textsc{mc}^{\mathrm{p}}_n \vdash^{m,k}_{r,l} \Gamma$, 92
$\mathrm{card}(\Gamma)$, 97
PBT, 101
$F^{\bar{*}}$, 101
$\textsc{gb}(l)$, 102
Isc_l, $M_l(n)$, $Z_l(X)$, $h(l)$, 103
$d_0 \sqsubseteq^l d_1$, 103
$\textsc{qb}(\vec{a})$, 104
$\mathrm{N}_d(F)$, $\mathrm{N}_d(\Delta)$, 105
$\mathrm{R}^m(N)$, $\mathrm{R}^m_l(N)$, 106
$\mathrm{A}(N)$, $\mathrm{A}^m(N)$, 107
DOA, 111
$f \leq g$, $\mathcal{H}(F)$, 112
$\mathcal{DO}(T)$, 112
$T_1 \subseteq T_2$, $T_1 \subsetneq T_2$, 116
$\mathcal{DO}(T)$, 118
$F^{el_{\mathcal{F}^i}}$, 123

Key–words

${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Inversion, 99
$(\bigvee)$-Exportation, 28, 91, 92
$(\bigwedge)$-Inversion, 28, 91, 92
$(\forall)$-Inversion, 91, 92
Cantor normal form, 20
Gödel numbers, 45

accessibility operator, 37

bounded
  formula, 2, 60
  normal b. formula, 60
  normal b. quantifier, 60
  quantifier, 18, 60
  sharply b. formula, 60
  sharply b. quantifier, 60
bounded arithmetic, 2
bounded predicative arithmetic, 8, 58
Boundedness
  Lemma, 39
    Predicative, 107
  Theorem, 40
    Predicative, 101

collection schema
  sharply bounded, 69
course-function, 48
cut-elimination
  partial, 64

Dynamic Ordinal, 5, 9, 112
  analysis, 112
dynamic predicative truth complexity, 92

Elimination
  Lemma, 29
    $\mathcal{L}_\omega$-Cut-E. L., 35
    ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Cut-E. L., 99
    Extended, 97
  Theorem, 29
    $\mathcal{L}_\omega$-Cut-E. T., 36
    ${}^{\mathrm{p}}\Delta_0^{\mathrm{b}}(\mathcal{X})^{*_p}$-Cut-E. T., 99
    Extended, 98
Embedding Theorem, 93
enumeration function, 38
  dual, 38
Equality Lemma, 28, 91, 92
exponential notation
  $\mathcal{E}$ and $\prec$, 51
  $\mathrm{T}_{\mathcal{E}}$, 55
  $\hat{+}$, 54
  predecessor, 56
  set of, $\mathcal{E}$, 47
  successor, 52
exponentiation
  function on $\mathcal{E}$, 47

formula
  $\Sigma_1^0(\mathcal{X})$, 4
  arithmetical, 18
  atomic, 18, 25
  bounded, 2, 60
  first order, 18
  infinitary, 25
  normal, 60
  of $\mathcal{L}_{BPA}$, 59
  of $\mathcal{L}_{Z_1}$, 18
  predicative bounded, 60
  predicative ground, 60
  prime, 18, 25
  sharply bounded, 60

good measurement, 4, 111

heriditary inversion, 34
hull
  $\leq$-, 112
  $\leq$-, 5

impredicative
  individual, 8
  variables, 58
indiscernibles, 9, 103
induction axioms, 19, 62
instance of a formula, 63

jump of X, 22, 79

language
  $\mathcal{L}_{BPA}(\mathcal{X})$, 59
  $\mathcal{L}_{BA}$, $\mathcal{L}_{BA}(\mathcal{X})$, 67
  $\mathcal{L}_{BPA}$, 58
  $\mathcal{L}_{Z_1}$, 17
  $\mathcal{L}_\infty$, 25
  $\mathcal{L}^p_\omega$, 87
  in TAIT-style, 17
  infinitary, 25
length of F, 35
limited recursion, 15
  course-of-values, 48
  on notation, 15

majorized
  f is majorized by g, 5, 112
minimization axioms, 119
monotone polynomial, 15
Monotonicity Lemma, 37, 105
multi-cut, 92

natural numbers, 13
negative points, 37, 105
normal
  derivation, 64
  term, 59
  variables, 59

order-type, 19
ordinal analysis, 3, 17

polynomial growth rate, 15, 45
polynomial time hierarchy, 1
polynomially bounded, 8
polytime functions, 13, 14
predicative
  bounded formula, 60
  ground, 60
  individual, 8
  length, 89
  rank, 88
  term, 59
  truth complexity, 91
  variables, 58
prenex, 2, 68
proof-theoretical ordinal, 3, 19
pure number theory, 17

quantifier
  bounded, 18, 60
  normal bounded, 60
  sharply bounded, 60
  unbounded, 18

rank
  of an $\mathcal{L}_\infty$-formula F, 26
  predicative, 88
reachability operator, 38, 106

safe
  variables, 59
selection sequences, 34
sentences
  Π^1_1-sentences of $\mathcal{L}_{Z_1}$, 18
  of $\mathcal{L}_{Z_1}$, 18
set

- `Seq` of Gödel numbers, 46
- variables, 59

strict, 2, 68

Structural Rule, 28, 91, 92

Substitution Rule, 91, 92

system
- auxiliary infinitary, 28
- formal, 27
- infinitary, 27
- predicative bounded semi-formal s., 90

term, 59
- ground, 18
- normal, 59
- of $\mathcal{L}_{Z_1}$, 18
- predicative, 59
- predicative ground, 60

transformation
- $F^{el_{\mathcal{F}^i}}$, 78, 123

translation
- F^*, 26
- F^{*_p}, 88
- $F^{\setminus\!\!\!*}$, 101

truth complexity, 27
- dynamic predicative, 92
- predicative, 91

unbounded
- quantifier, 18

variables
- impredicative, 58
- normal, 59
- number, 17
- predicative, 58
- safe, 59
- set, 17, 59

wellfoundedness
- notion of w., 19