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Deciding logics of linear Kripke frames with scattered end pieces

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Abstract We show that logics based on linear Kripke frames – with or without constant domains – that have a scattered end piece are not recursively enumerable. This is done by reduction to validity in all finite classical models.

1 Introduction

This article combines and extends (and fixes some errors) of [3] giving non recursive enumerability results for Gödel logics, and [4], linking Gödel logics with logics defined by countable Kripke frames. The extensions presented here are two-fold:

- extension to linear Kripke frames of arbitrary size,
- extension to the case of increasing domains.

Similar studies have been carried out by Takano [8], who provided an axiomatization of the Kripke frame of $\mathbb{R}$, and Minari et al. [5] who explored relations between logics of Kripke frames based on ordinals.

2 Preliminaries

Definition 1 (Kripke frame) A *Kripke frame* is a triple $(K, R, \mathcal{U})$ where (K, R) is a quasi order (reflexive and transitive), and $\mathcal{U}$ is a mapping that associates with each $k \in K$ a non-empty set of objects U_k with the following conditions: if $k\ R\ l$ then $U_k \subseteq U_l$.

If $\mathcal{U}$ is the constant function, then it is a Kripke frame *over constant domains*, otherwise normal, general, or *increasing domains*.

If R is a total order, then we say that the Kripke frame is *linear*.

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Henceforth, we will only deal with linear Kripke frames, and will drop "linear" at will.

In standard Kripke style semantics evaluations are considered via forcing relations in the worlds, together with conditions that guarantee persistency, i.e., that if A holds in a world w, it also holds in all worlds w' such that $w\ R\ w'$. Instead of following this approach we will consider the set of upward closed subsets as co-domain of valuations. This definition of semantics will be equivalent to the approach via forcing relations.

Definition 2 (upsets) The set $\mathrm{Up}(K)$ consists of all upward closed subsets of K.

In case of linear Kripke frames the set $\mathrm{Up}(K)$ is a complete total order with respect to $\subseteq$, i.e. a complete lattice. We will denote with $\mathbf{0_K}$ the empty set, $\mathbf{0_K} = \emptyset$, and with $\mathbf{1_K}$ the full frame, $\mathbf{1_K} = K$. Clearly, $\mathbf{0_K}$ is the smallest element, and $\mathbf{1_K}$ the largest element of this lattice.

We will furthermore freely use notations from linear orders, especially intervals like $[a, b]$ for a and b in $\mathrm{Up}(K)$, with the usual meaning $[a, b] = \{c \in \mathrm{Up}(K) : a \subseteq c \subseteq b\}$.

We will base our definition of semantics on $\mathrm{Up}(K)$ instead of K itself. This is straight-forward in the constant domain case, but in the increasing domain case we have to take care that the evaluations honor the persistency condition mentioned above.

Definition 3 (valuation) Let $\bar{U}$ be the union of all U_k for $k \in K$. A function φ is called *valuation* if it is mapping atomic formulas with constant symbols for all $u \in \bar{U}$ Into $\mathrm{Up}(K)$, such that for an n-ary predicate symbol P

$$\forall w \in \varphi(P(\mathbf{u})) : \mathbf{u} \in U_w^n$$

Let $\mathcal{L}_{\bar{U}}$ be the set of all formulas with with constant symbols from $\bar{U}$. It is easy to show that the above condition ensures that the persistency condition holds for all atomic formulas. In addition, φ can be extended to a valuation of all formulas in $\mathcal{L}_{\bar{U}}$ into $\mathrm{Up}(K)$ in the following way:

$$\begin{aligned}
\varphi(A \vee B) &= \varphi(A) \cup \varphi(B) \\
\varphi(A \wedge B) &= \varphi(A) \cap \varphi(B) \\
\varphi(A \to B) &= \begin{cases} \mathbf{1_K} & \text{if } \varphi(A) \subseteq \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases}
\end{aligned}$$

To extend the definition of valuation to quantifiers we introduce an auxiliary existance predicate: Let $E(x)$ be a new unary predicate symbol, whose interpretation in K is fixed according to K's domains:

$$w \in \varphi(E(a)) \quad \text{iff} \quad a \in U_w$$

for all $w \in K$ and $a \in \bar{U}$. With this we define

$$\begin{aligned}
\varphi(\exists x A(x)) &= \bigcup_{a \in \bar{U}} \varphi(E(a) \wedge A(a)) \\
\varphi(\forall x A(x)) &= \bigcap_{a \in \bar{U}} \varphi(E(a) \to A(a))
\end{aligned}$$

Similar approaches have been taken by Scott [7] and are used for Skolemization by Baaz and Iemhoff [1,2].

We are considering valuations of formulas with parameters from all worlds $\bar{U}$. It is easy to show that a formula can only we evaluated to true in such worlds in which all its parameters are present:

Proposition 1 *Let φ be a valuation, $w \in K$, $A \in \mathcal{L}_{\bar{U}}$ and $a \in \bar{U}$ a parameter occurring in A. Assume $w \in \varphi(A)$. Then $a \in U_w$.*

The above definition of semantics is equivalent to the more common definition via a forcing relation: Let φ be a valuation, $w \in K$ and $A \in \mathcal{L}_{\bar{U}}$. We define $w \Vdash A$ inductively as follows: If A is atomic then $w \Vdash A$ iff $w \in \varphi(A)$.

$w \Vdash A \vee B$	iff	$w \Vdash A$ or $w \Vdash B$.
$w \Vdash A \wedge B$	iff	$w \Vdash A$ and $w \Vdash B$.
$w \Vdash A \to B$	iff	for all $u \in K$ such that $w\, R\, u, u \Vdash A$ implies $u \Vdash B$.
$w \Vdash \exists x A(x)$	iff	there exists $a \in U_w$ such that $w \Vdash A(a)$.
$w \Vdash \forall x A(x)$	iff	for every $u \in K$ such that $w\, R\, u$, and every $a \in U_u$, $u \Vdash A(a)$.

Proposition 2 *The two definitions of semantics are equivalent. That is, let φ be a valuation, $w \in K$ and $A \in \mathcal{L}_{\bar{U}}$. Then*

$$w \Vdash A \quad \textit{iff} \quad w \in \varphi(A).$$

Proof by induction on the build-up of A. □

Definition 4 (validity) The logic $L(K)$ of a fixed Kripke frame K is the set of all closed formulas such that for all valuations φ the valuation of A is $\mathbf{1_K}$

At times we will use the notation $\varphi \models A$ if $\varphi(A) = \mathbf{1_K}$. In the following we will give two related but independent proofs for the case where the Kripke frame is scattered or not.

Definition 5 A Kripke frame is called scattered if $\mathbb{Q}$ cannot be embedded into the Kripke frame as a linear order.

As a fact, K is scattered if and only if $\mathrm{Up}(K)$ is scattered: an embedding of $\mathbb{Q}$ into K easily carries over to an embedding in $\mathrm{Up}(K)$, and conversely, given an embedding into $\mathrm{Up}(K)$, one can construct a dense in itself subset of K, by iteratively choosing levels of intermediate points of previous levels, together with separating upsets.

In the logics of Kripke frames, not all quantifier shift rules are valid. To obtain validity for some formulas, we consider 'crisp' formulas which behave classically. This will be used in the following during our construction to give strict (crisp) meaning to certain axioms.

Definition 6 A formula is called *crisp* if all occurrences of atomic formulas are negated.

Lemma 1 *Let A and B be crisp formulas, K a Kripke frame. If A and B are classically equivalent, then $A \leftrightarrow B$ is valid in K, i.e. $A \leftrightarrow B \in L(K)$. Moreover, if $A(x)$ and B are crisp, then*

$$\models \forall x A(x) \to B \leftrightarrow \exists x(A(x) \to B) \quad \textit{and}$$
$$\models B \to \exists x A(x) \leftrightarrow \exists x(B \to A(x)).$$

Proof By induction on the formula structure: Atomic formulas are either negated or double-negated, and thus obtain as possible values only $\mathbf{0_K}$ or $\mathbf{1_K}$. Gödel valuations are truth-functional, thus compound formulas, too, obtain only the extremal truth values.

Assume that $A \leftrightarrow B$ is not valid in $L(K)$, by the above we can assume w.l.o.g. that $\varphi_K(A) = 0$ and $\varphi_K(B) = 1$. Define a classical interpretation with universe being the objects of the initial world, and $\varphi_C(P(a))$ holds iff $\varphi_K(\neg\neg P(a)) = \mathbf{1_K}$. We obtain that also φ_C disagrees on the values of A and B, and thus $A \leftrightarrow B$ is not classically valid.

While not completely reflecting the strict order, the following definition comes very close.

Definition 7 We write $A \prec B$ for $(B \to A) \to B$.

$A \prec B$ evaluates to $\mathbf{1_K}$ iff $\varphi(A) \subsetneq \varphi(B)$, or $\varphi(A) = \varphi(B) = \mathbf{1_K}$. Thus, with the exception of $\mathbf{1_K}$, $\prec$ behaves like a strict order.

In the following sections we will show that various subclasses of Kripke frames generate a non-recursively enumerable logic, by reducing classical validity of a formula in all finite models to the validity of a formula in the logic under discussion. Since validity in all finite first-order classical structures is not recursively enumerable, we obtain that the logics under consideration are not recursively enumerable.

3 Scattered linear Kripke frames over constant domains

Following the layout described above, we are searching for something genuinely finite in a scattered Kripke frame. The property of $\mathbb{Q}$ that is exploited here is that for every two distinct elements of $\mathbb{Q}$, there is another element strictly between these two. On the contrary, if we cannot embed $\mathbb{Q}$, i.e., if the Kripke frame is scattered, then there are two elements with no other world in-between. We can find these two elements by iterating a picking procedure: Start with two arbitrary distinct elements, they form level 0; at each stage select two elements from previous levels, and search for an element strictly between them. If it exists, select one for the next level, otherwise terminate the process.

This process can be continued ad infinitum only if the starting set was dense in itself, that is contains a copy of $\mathbb{Q}$. If this is not the case, the process will terminate at a *finite* stage, for any choice of intermediate points.. By relativizing all quantifiers to indices of non-empty levels we are quantifying over arbitrary but finite domains.

For the reverse direction (translating validity from Kripke logics to classical logic) we need to make sure that we can find a picking order in a way that the process iterates to an arbitrary but finite number. This can be achieved in the case the original Kripke frame was infinite by selecting the initial points with sufficiently many points in-between $(2^n + 1)$.

Theorem 1 *If K is scattered and infinite, then $L(K)$ is not recursively enumerable.*

Proof We associate to every sentence A another sentence A^+ in a way that A^+ is valid in $L(K)$ iff A is true in every finite (classical) first-order structure. Note that the underlying languages are the same.

To formalize the procedure laid out above we use the following notions and special predicate symbols, where we assume that the underlying language contains infinitely many predicate symbols for any arity:

Level definition We select a binary predicate L that divides a subset of the domain into levels. We write $x \in i$ as an abbreviation for $\neg\neg L(x, i)$. Note that this formula is crisp, i.e., evaluates to $\mathbf{0_K}$ or $\mathbf{1_K}$.

Order We select a unary predicate P that acts as 'valuator' in the sense that when we compare variables x and y, we actually compare the valuations of $P(x)$ and $P(y)$. That is, we write

$$x \prec y \quad \leftrightarrow \quad (P(x) \prec P(y)).$$

Recall that if $\varphi(P(x)) \neq \mathbf{1_K}$, $x \prec y$ evaluates to $\mathbf{1_K}$ iff $\varphi(P(x)) \subset \varphi(P(y))$. In particular, if $\varphi(\exists x P(x)) \neq \mathbf{1_K}$, this always holds.

Emptiness predicate Let $R(i)$ express the fact that the level i is not empty, i.e., $R(i) \leftrightarrow \exists x(x \in i)$.

The general layout of our formula A^+ is

$$A^+ = \mathit{Construction} \rightarrow (A' \vee \exists x P(x)) \tag{1}$$

The ingredients of A^+ are:

Construction A formula realizing the picking process. Besides the actual picking process, we also need to add the (finite) set of standard axioms for Robinson's Q [6].

A' The original formula A with two changes: (i) Putting double negations in front of every atomic formula to obtain a crisp formula; and (ii) relativizing all quantifiers to the predicate $R(i)$.

$\exists x P(x)$ This formula is necessary to ensure that when translating counter examples, the $\prec$ relation provides an actual strict relation.

Let us now turn our attention to the *Construction* part: It will contain the following formulae:

- crispified (double negations in front of atomic predicates) (finite) set of standard axioms for Robinson's Q for 0, successor, and the less-than-or-equal relation. In the following we will denote with "$\leq$" the formula obtained by double-negation of the less-than-or-equal relation.
- assumption that the level 0 is not empty and contains two different objects
- an assumption that if level j is not empty, then all previous levels are also not empty. This is necessary to deal with non-standard domains of models of Q.
- the description of the process: let

$$D \equiv \begin{array}{c}(j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \rightarrow \\ \rightarrow (z \in s(i) \wedge x \prec z \wedge z \prec y)\end{array}$$

which will be used to express that for any two elements $x \prec y$ of levels $\leq i$ there is an element between x and y in the next level $i+1$.

We use this formula to state that either the above mentioned picking process can be done, or the following level is empty:

$$\forall i\big[\forall x, y \forall j \forall k \exists z\, D \lor \forall x \neg(x \in s(i))\big]$$

Combining these we obtain the following formula

$$\mathit{Construction} = \left\{ \begin{array}{l} \mathsf{Q} \land c_1 \in 0 \land c_2 \in 0 \land c_2 \prec c_1 \\ \quad \land \forall i, j(i \leq j \land R(j) \to R(i)) \\ \quad \land \forall i\big[\forall x, y \forall j \forall k \exists z\, D \lor \forall x \neg(x \in s(i))\big] \end{array} \right\} \tag{2}$$

If *Construction* is true, then the true Q axioms force the domain to be a model of arithmetic, which could be either a standard model (isomorphic to $\mathbb{N}$) or a non-standard model ($\mathbb{N}$ followed by copies of $\mathbb{Z}$). P orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

As laid out above, if we cannot continue the picking process at infinitum, the levels above some i are empty. Clearly, this condition can be satisfied in an interpretation φ only for finitely many levels if $\mathrm{Up}(K)$ does not contain a dense subset, i.e., K is scattered, since if infinitely many levels are non-empty, then $\bigcup_i \{\varphi(P(d)) : \varphi \models d \in i\}$ gives a dense subset of $\mathrm{Up}(K)$. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain.

In the following we show that validity of A in all finite classical models coincides with validity of A^+ in $L(K)$ by translating counter-examples. The following Lemma 2 is the easy direction where we translate a counter example in a finite structure into a counter example in Kripke frames. The later Lemma 3 is the core of the proof, as it translates a counter example from Kripke frames to finite structures.

Lemma 2 *If A is classically false in some finite structure φ_C, then there is a model φ_K based on constant domains in which A^+ does not hold.*

Proof Suppose A is classically false in some finite structure φ_C. W.l.o.g. we may assume that the domain of this structure are the natural numbers $0, \ldots, n$. We extend φ_C to an $\mathrm{Up}(K)$-interpretation φ_K with constant domain $\mathbb{N}$ as follows: Since $\mathrm{Up}(K)$ contains infinitely many subsets (we assumed that K is infinite), we can choose c_1, c_2, L and P so that $\exists x(x \in i)$ is true for $i = 0, \ldots, n$ and false otherwise, and so that $\varphi_K(\exists x\, P(x)) \neq \mathbf{1_K}$.

One way to do this is to start with two points c_1 and c_2 such that there are at least 2^{n+1} points between them. By reusing already selected points as far as possible, we need $\sum_{i=0}^{n} 2^n = 2^{n+1} - 1$ points. Trivial observation shows that in this case we can always assign values to $\varphi_K(L(x,i))$ in a way that the picking process can continue exactly to the n-th stage, and breaks exactly at stage $n+1$ from where on all levels are empty.

The number-theoretic symbols receive their natural interpretation. By this we see that *Construction*, the antecedent of A^+, receives the valuation $\mathbf{1_K}$, and the consequent receives $\varphi_K(\exists x\, P(x)) \neq \mathbf{1_K}$, so $\varphi^+ \not\models A^+$. Since every model of constant domain is also a model of increasing domain, this direction is shown.

(End of proof of Lemma 2)

We continue with the reverse direction where we want to show that if φ_K is a counter model of A^+, we can find a counter model of A in a finite structure. To this end we start with showing the following fact:

Fact If φ_K is a counter model for A^+, then $\varphi_K(\mathit{Construction}) = \mathbf{1_K}$.

Proof Suppose that $\varphi_K \not\models A^+$, that is that

$$\varphi_K(\mathit{Construction} \to (A' \vee \exists x P(x))) < \mathbf{1_K}.$$

Thus, both $\varphi_K(A')$ and $\varphi(\exists x P(x))$ are $< \mathbf{1_K}$. In this case, $\varphi_K(x \prec y) = \mathbf{1_K}$ iff $\varphi_K(P(x)) \subset \varphi_K(P(y))$, so $\prec$ defines a strict order.

Next, let us inspect the possible valuations of sub-formulas of *Construction*:

- Q is a crisp formula, thus evaluates to $\mathbf{0_K}$ or $\mathbf{1_K}$;
- formulas with $\in$ or $\leq$ as main predicates are also crisp − this includes also those containing $R(i)$; remember that $\leq$ denotes the double-negated less-than-or-equal relation;
- the remaining formulas in D are of the form $x \prec y$, which is a shorthand for $P(x) \prec P(y)$.

Thus, all the sub-formulas evaluate either crisp, i.e., to $\mathbf{0_K}$ or $\mathbf{1_K}$, or are evaluated $\leq \varphi_K(\exists x\, P(x))$, which in turn guarantees that the whole *Construction* also evaluates to either $\mathbf{1_K}$ or $\leq \varphi_K(\exists x\, P(x))$. Since we assume that φ_K is a counter model (the antecedent needs to be greater than the succedent), it must be $\mathbf{1_K}$. □

Lemma 3 *If φ_K is a countermodel of $A+$, then there is a finite structure φ_C such that A does not hold in φ_C.*

Proof By the previous fact we see that $\varphi_K(\mathit{Construction}) = \mathbf{1_K}$, and as a consequence we obtain that *Construction* actually codifies the picking process as described, i.e., that $x \in i$ defines a series of levels and any level $i > 0$ is either empty, or for all x, and y occurring in some smaller level there is a z with $x \prec z \prec y$ and $z \in i$.

Due to the fact that we are considering increasing domains, we cannot proceed directly to the definition of the counter model φ_C, but have to base the model on the set of objects that are present in *all* worlds. Let U^+ be the set of objects that are present in all universes U_k, i.e. $U^+ = \bigcap_{k \in K} U_k$. Due to the presence of the Q axioms that are evaluated under φ_K to $\mathbf{1_K}$, U^+ must contain a copy of $\mathbb{N}$. For $i \in \mathbb{N}$, let L_i be the set of all elements of U^+ of levels i:

$$L_i := \left\{ u \in U^+ : \varphi_K(u \in i) = \mathbf{1_K} \right\}$$

and let $L_{\leq i} := \bigcup_{j \leq i} L_j$.

The next fact shows that the picking process is faithful, in the sense that levels are non-empty only if for any two elements of the previous levels an element in-between can be found, and otherwise empty:

Fact For a counter model φ_K of A^+, and all $i \in \mathbb{N}$, one of the following two statements hold:

1. $L_{i+1} = \emptyset$
2. For all x and y from $L_{\leq i}$ such that $\varphi_K(x \prec y) = \mathbf{1_K}$, there is a $z \in L_{i+1}$ such that $\varphi_K(x \prec z \prec y) = \mathbf{1_K}$

Proof Consider the relevant part of *Construction*,

$$B = \forall i \big[\forall x, y \forall j \forall k \exists z\, D \vee \forall x \neg (x \in s(i)) \big].$$

If $\varphi_K(B) = \mathbf{1_K}$, then for all $i \in U^+$, either $\varphi_K(\forall x, y \forall j \forall k \exists z\, D) = \mathbf{1_K}$ or $\varphi_K(\forall x \neg (x \in s(i))) = \mathbf{1_K}$.

Case $\varphi(\forall x, y\, \forall j \forall k \exists z\, D) = \mathbf{1_K}$*:* Note that this holds for all x, y, j, and k in U^+.

Now suppose that for all $z \in U^+$, $\varphi(D) \neq \mathbf{1_K}$, yet $\varphi_K(\exists z\, D) = \mathbf{1_K}$. This requires a sequence of $v \in U^+$ such that $\varphi_K(v, D)$ converges to $\mathbf{1_K}$. Then for at least some z that also has to be in U^+, the value of that formula would have to be $\supset \varphi_K(\exists z\, P(z))$, which is impossible. Thus, for every x, y, j, k in U^+, there is some $z \in U^+$ such that $\varphi_K(D) = \mathbf{1_K}$. But this means that for all x, y s.t. $x \in j$, $y \in k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in L_{i+1}$.

Case $\varphi(\forall x \neg (x \in i)) = \mathbf{1_K}$*:* In this case we immediately obtain that $\varphi_K(\neg(x \in i)) = \mathbf{1_K}$ for all $x \in U^+$, hence $\varphi_K(x \in i) = \mathbf{0_K}$ and L_i is empty.

(End of proof of Fact)

Assume that at least the first ω level sets are non-empty, and consider $L_\omega := \bigcup_{i<\omega} L_i$. L_ω is dense wrt $\prec^{\mathcal{M}}$, hence $\{\varphi_C(P(u)) : u \in L_\omega\}$ is dense wrt $\subset$, and we obtain a nontrivial dense subset of $\mathrm{Up}(K)$ which cannot exists.

As a consequence, we obtain that for some i in the (possible non-standard) model of Q, the level i is empty, i.e., $L_i = \emptyset$.

Let i_0 be minimal with $L_{i_0} = \emptyset$. Then the additional assumption in the antecedent of A^+ shows for $i \in U^+$

$$\varphi_K(R(i)) = \mathbf{1_K} \quad \Rightarrow \quad \varphi_K(i < i_0) = \mathbf{1_K} \quad \Rightarrow \quad i < i_0$$

Note that since there are only finitely many non-empty levels, the first non-empty level i_0 needs to be element of the standard part of a (possibly non-standard) model. Thus, A is false in the classical interpretation φ_C obtained from φ_K by restricting φ_K to the domain $\{i : i < i_0\}$ and $\varphi_C(Q) = \varphi_K(\neg\neg Q)$ for atomic Q, i.e., A is false in the finite model $(\{0, \ldots, i_0 - 1\})$ by letting

$$\varphi_C(Q) = \text{ true} \qquad \Leftrightarrow \qquad \varphi_K(\neg\neg Q) = \mathbf{1_K}$$

The very same proof applies to both increasing and constant domain case, which proves the theorem.

(End of proof of Lemma 3)

The combination of Lemma 2 and 3 provides a proof of Theorem 1. □

As a consequence we obtain for example that for every ordinal, the logic of the Kripke frame based on it is not recursively enumerable.

4 Not-scattered but scattered end piece

Considering the case where the Kripke frame is not scattered, we know that there are axiomatizable logics, e.g., $L(\mathbb{Q})$ or $L(\mathbb{R})$ [8]. But we can show that if the Kripke frame has a scattered end piece, or equivalently the complete linear order $\mathrm{Up}(K)$ has a scattered initial segment, a proof similar to the preceeding section can exhibit that logics for such Kripke frames also are non-recursive enumerable.

If the linear order $\mathrm{Up}(K)$ has a scattered initial segment, then for any sequence of $(a_n)_{n\in\mathbb{N}}$ with limes equal $\mathbf{0_K} = \emptyset$, and for sufficiently large n, the initial segment $[\mathbf{0_K}, a_n]$ is scattered.

As with the proof of Theorem 1, we formalize a picking process. But depending on the starting elements we cannot be sure that it actually terminates after finitely many steps. Thus, we adjust the procedure by parallelizing the previous picking process in all intervals $[\mathbf{0_K}, a_n]$ for some decreasing sequence. In some of these intervals the picking process might continue ad infinitum, but due to the end piece of the Kripke frame being scattered, for sufficiently large N, the picking process in *all* intervals $[\mathbf{0_K}, a_n]$ for $n > N$ will terminate. By guaranteeing that if for one interval a certain level is empty, then for all intervals this level (and all following) are empty, we obtain a finite set of not empty levels to which we relativize the the formula.

Theorem 2 *If for an infinite Kripke frame K, K has an initial segment that is scattered, then $L(K)$ is not recursively enumerable.*

Proof The definition of A^* mirrors the definition of A^+ in the proof of Theorem 1, except that the construction there is carried out infinitely many times for $[\mathbf{0_K}, a_n]$ (in $\mathrm{Up}(K)$), where $(a_n)_{n\in\mathbb{N}}$ is a strictly descending sequence, $a_n > \mathbf{0_K}$ for all n, which converges to $\mathbf{0_K}$ in the linear order $\mathrm{Up}(K)$.

The basic concepts of the previous proof remain unchanged (level definitions, order, emptiness predicate), with one change: All the involved predicates (P, R, L) obtain another position referring to the current interval in which the construction is carried out. Thus we have:

Level definition We select a *ternary* predicate L, and write $x \in_\ell i \equiv \neg\neg L(x, i, \ell)$.

Order We select a *binary* predicate P that acts as valuator in the current interval. We write

$$x \prec_\ell y \equiv (P(y, \ell) \to P(x, \ell)) \to P(y, \ell).$$

Emptiness predicate Let $R(i, \ell)$ express the fact that the level i for the interval ℓ is not empty, i.e., $R(i, \ell) = \exists x(x \in_\ell i)$. The formula we will use to relativize is defined as $R^*(i) = \forall \ell R(i, \ell)$.

To formalize the decreasing sequence of intervals, we introduce a new concept:

Interval definition We select a unary predicate symbol $Q(\ell)$ to define the decreasing sequence of intervals. Note that $\varphi_K(\neg\forall\ell\, Q(\ell)) = \mathbf{1_K}$ iff $\inf\{\varphi_K(Q(d)) : d \in U^+\} = \mathbf{0_K}$ and $\varphi_K(\forall\ell\,\neg\neg Q(\ell)) = \mathbf{1_K}$ iff $\mathbf{0_K} \notin \{\varphi_K(Q(d)) : d \in U^+\}$.

Note that as before, for a fixed ℓ, provided $\varphi_k(\exists x\, P(x, \ell)) < \mathbf{1_K}$, $\varphi_K(x \prec_\ell y) = \mathbf{1_K}$ iff $\varphi_K(P(x, \ell)) < \varphi_K(P(y, \ell))$, and $\varphi_K(x \in_\ell y)$ is always either $\mathbf{0_K}$ or $\mathbf{1_K}$.

The general layout is again that we codify the construction in a formula *Construction*, and define A^* as:

$$\textit{Construction} \to (A' \vee \exists\ell\exists x\, P(x, \ell) \vee \exists\ell Q(\ell)) \qquad (3)$$

The ingredients of A^* are:

Construction The parallelized picking process, see below.

A' The original formula A with two changes: (i) Putting double negations in front of every atomic formula to obtain a crisp formula; and (ii) relativizing all quantifiers to the predicate $R^*(\ell)$ which states that *all* levels in the interval ℓ are not empty.

$\exists\ell\exists x P(x,\ell)$, $\exists\ell Q(\ell)$ These formulae are again necessary to ensure that when translating counter examples, the $\prec$ relation provides an actual strict relation.

The list of assumptions going into *Construction* mirrors the list from the previous proof, but contains in addition the necessary properties to force $Q(\ell)$ to have a limit at 0. The important part of the construction formula is in the last line, where for all levels we assume that either we can pick a new element (left part involving E), or for all intervals (ℓ) the level set is empty.

- Q (with double negations in front of atomic formulas) as before
- an assumptions that $Q(\ell)$ describes a strictly descending sequence to $\mathbf{0_K}$, with limes $\mathbf{0_K}$ but non of the elements being $\mathbf{0_K}$
- an assumption that all evaluations for interval ℓ are happening in the interval $[\mathbf{0_K}, \varphi_K(Q(\ell))]$
- an assumption that level 0 for each interval ℓ contains two different objects
- an assumption that if for interval ℓ level j is not empty, then previous levels for this interval are also not empty. This is necessary to deal with non-standard models of Q
- the description of the process as before, but with additional quantifier for all stages and relativized to these stages:

$$E \equiv \begin{array}{l}(j \leq i \wedge x \in_\ell j \wedge k \leq i \wedge y \in_\ell k \wedge x \prec_\ell y) \rightarrow \\ \quad \rightarrow (z \in_\ell s(i) \wedge x \prec_\ell z \wedge z \prec_\ell y)\end{array}$$

Combining the above items we obtain:

$$\mathit{Construction} = \left\{ \begin{array}{l} \mathsf{Q} \wedge \neg\forall\ell\, Q(\ell) \wedge \forall\ell\, \neg\neg Q(\ell) \wedge \\ \quad \forall\ell\forall x(P(x,\ell) \prec Q(\ell)) \wedge \\ \quad \forall\ell\forall i,j(i \leq j \wedge R(j,\ell) \rightarrow R(i,\ell)) \wedge \\ \quad \forall\ell\exists x\exists y(x \in_\ell 0 \wedge y \in_\ell 0 \wedge x \prec_\ell y) \wedge \\ \quad \forall i\big[\forall\ell\forall x,y\forall j\forall k\exists z\, E \vee \forall x\forall\ell\neg(x \in_\ell s(i))\big] \end{array} \right\} \tag{4}$$

Recall from the proof of the former theorem that we collect all objects that are present in domains at all worlds in U^+. The idea here is that an interpretation φ_K will contain a sequence $(a_n)_{n\in\mathbb{N}} \to \mathbf{0_K}$ given by $a_n = \varphi_K(Q(\ell_n))$ for some elements ℓ_n in the domain. We have $a_n > a_{n+1}$ (which is $a_n \supset a_{n+1}$), and $\mathbf{0_K} < a_n < \mathbf{1_K}$ for all n. Let $L^i_\ell = \{x \in U^+ : \varphi_K(x \in_\ell i)\}$ be the i-th level for interval ℓ. $P(x,\ell_n)$ orders the set $\bigcup_i L^i_{\ell_n} = \{x \in U^+ : \varphi(\exists i(x \in_{\ell_n} i)) = \mathbf{1_K}\}$ in a sub-ordering of $[\mathbf{0_K}, a_n]$: $x \prec_{\ell_n} y$ iff $\varphi(x \prec_{\ell_n} y) = \mathbf{1_K}$. Again we force that whenever $x, y \in L^i_\ell$ with $x \prec_\ell y$, there is a $z \in L^{i+1}_\ell$ with $x \prec_\ell z \prec_\ell y$, or, if no possible such z exists, $L^{i+1}_\ell = \emptyset$. In addition, in this case also the level sets for all other intervals are empty, $L^{i+1}_{\ell'} = \emptyset$ for all ℓ.

Now if A is classically false in some finite structure φ_C, we can again choose a $\mathrm{Up}(K)$-interpretation φ_K in which the interpretations of P, Q, L are as intended, the number theoretic predicates and functions receive their standard interpretation, and the interpretation works in every interval as in the previous proof.

On the other hand, if $\varphi \not\models A^*$, then the value of the consequent is $< \mathbf{1_K}$. Then as required, for all x, ℓ, $\varphi(P(x,\ell)) < \mathbf{1_K}$ and $\varphi_K(Q(\ell)) < \mathbf{1_K}$. Since the antecedent, as before, must be $= \mathbf{1_K}$, this means that $x \prec_\ell y$ expresses a strict ordering of the elements of L^i_ℓ. and $\varphi_K(Q(\ell))$ for ℓ contains a strictly descreasing sequence towards $\mathbf{0_K}$. As K has scattered end-piece, the interval $[\mathbf{0_K}, \varphi_K(Q(\ell))]$ must be scattered for some ℓ. For this interval, some level set must be empty, thus the same level is also empty for all other intervals. The other conditions are likewise seen to hold as intended, so that we can extract a finite counter model for A based on the interpretation of the predicate symbols of A on $\{i : \varphi_K(R(i)) = \mathbf{1_K}\}$, which must be finite. □

5 Conclusions

This note shows that the method originally employed in [3] can successfully be extended to a much larger class. In fact, forthcoming work will extend this to logics with the △-operator, an indicator operator for $\mathbf{1_K}$.

This leads to the question whether it might be possible to extended the results to the case of non-linear Kripke frames. At least if we can restrict the branching degree it might be possible to show that if all branches in a (non-linear) Kripke frame are scattered, then the respective logic is not r.e.

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