A non-well-founded primitive recursive tree provably well-founded for co-r.e. sets

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Abstract

We construct by diagonalization a non-well-founded primitive recursive tree, which is well-founded for co-r.e. sets, provable in Σ^0_1-IND. It follows that the supremum of order-types of primitive recursive well-orderings, whose well-foundedness on co-r.e. sets is provable in Σ^0_1-IND, equals the limit of all recursive ordinals ω_1^{CK}.

This work contributes to the investigation of replacing the quantification over all sets in the definition of the proof-theoretic ordinal of a theory by a quantification over certain definable sets such as all arithmetical sets, or certain levels of the arithmetical hierarchy. We discuss this as a first step towards an investigation of replacing the set parameter in the definition of the dynamic ordinal of fragments of bounded arithmetic (cf. [2]) by certain definable sets in order to obtain non-relativized separation results.[1]

The proof-theoretic ordinal $\mathrm{O}(T)$ of a theory T can be defined by

$$\mathrm{O}(T) := \sup\{\alpha \ : \alpha \text{ is the order-type of a simple well-ordering } \prec \text{ and } T \vdash \mathrm{Wf}(\prec)\}$$

where "simple" means something suitable like Δ_0 or primitive recursive (prim. rec.) or recursive, depending on the theories under consideration, and $\mathrm{Wf}(\prec)$ expresses the well-foundedness of $\prec$ by the Π^1_1 sentence

$$(\forall X)\,\mathrm{Found}(\prec, X) \equiv (\forall X)\Big[(\forall x)((\forall y \prec x)(y \in X) \rightarrow x \in X) \rightarrow (\forall x)(x \in X)\Big].$$

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[1] I like to thank A. Setzer for his hospitality during my stay in Uppsala in December 1998 – these investigations are inspired by a discussion with him; S. Buss for his hospitality during my stay at UCSD and for valuable remarks on a previous version of this paper; and M. Möllerfeld for remarks on a previous title.

From [5] and [6, pp. 280-284] follows the existence of an ordering $\prec_1$ which is not well-founded but PA-provably well-founded on arithmetical sets. I.e., if we take $\mathrm{Found}(\prec, \Pi^1_0)$ to be the schema of transfinite induction of $\prec$ for Π^1_0-sets (i.e. arithmetical sets), then the instances of $\mathrm{Found}(\prec_1, \Pi^1_0)$ are theorems of PA. From this it is not hard to show that if we replace $\mathrm{Wf}(\prec)$ in the definition of $\mathrm{O}(PA)$ by the schema $\mathrm{Found}(\prec, \Pi^1_0)$, then we get ω_1^{CK}:

$$\tilde{\mathrm{O}}(PA) := \sup\{\alpha : \alpha \text{ is the order-type of some recursive well-ordering } \prec \text{ and } PA \vdash \mathrm{Found}(\prec, \Pi^1_0)\} = \omega_1^{CK}.$$

We will repeat an argument given by Arai in [1] which shows this. It will be immediate that this result still holds if we restrict to prim. rec. well-ordering.

But what happens if we consider sub-theories of PA? Remember that we are heading towards fragments of bounded arithmetic. For a class Φ of formulas let $\mathrm{O}_\Phi(\Sigma^0_1\text{-IND})$ be the result of replacing $\mathrm{Wf}(\prec)$ in the definition of the proof-theoretic ordinal $\mathrm{O}(\Sigma^0_1\text{-IND})$ by the schema $\mathrm{Found}(\prec, \Phi)$. Our main result will be that in case of co-r.e. sets, i.e. $\Phi = \Pi^0_1$, we still get ω_1^{CK}:

Theorem 1.

$$\mathrm{O}_{\Pi^0_1}(\Sigma^0_1\textit{-IND}) = \sup\{\alpha : \alpha \textit{ is the ordertype of some prim. rec. well-ordering } \prec \textit{ and } \Sigma^0_1\textit{-IND} \vdash \mathrm{Found}(\prec, \Pi^0_1)\} = \omega_1^{CK}.$$

A direct adaption of the construction given in [6, pp. 280-284] would only prove the existence of a prim. rec, not well-founded ordering $\prec$ which would be Σ^0_2-IND-provably well-founded on Π^0_1 sets.

In our arguments we will consider only recursive ordinals which are given by well-founded recursive trees. Using the Kleene-Brouwer ordering we could obtain well-orderings from well-founded trees of even bigger ordertype, but this would unnecessarily make our arguments more complicated. Recall that a tree T is a subset of the set of all finite sequences of natural numbers $\omega^{<\omega}$ which is closed under initial subsequences. With $\prec_{tr}$ we denote the usual tree ordering, i.e. the converse of the strict initial subsequence ordering on $\omega^{<\omega}$. Let $\prec_T$ be the restriction of $\prec_{tr}$ to T, $\prec_T = \prec_{tr} \cap T^2$. With $\mathrm{Found}(T, A)$ we mean the formula $\mathrm{Found}(\prec_T, A)$. A tree T is called well-founded iff $\prec_T$ is well-founded, i.e. $\mathrm{Found}(\prec_T, A)$ holds for all sets A. Let $|T|$ denote the ordertype of a well-founded tree T, i.e. the ordertype of $\prec_T$, which is the same as the height of T. Let $\sigma ^\frown \tau$ denote σ concatenated with τ for $\sigma, \tau \in \omega^{<\omega}$, and let $T[\sigma]$ be the subtree of T starting at σ: $T[\sigma] = \{\tau : \sigma ^\frown \tau \in T\}$.

We repeat the argument given by Arai in [1] which shows that $\tilde{\mathrm{O}}(PA) = \omega_1^{CK}$. To this end, it is enough to show

Lemma 2. *For every recursive well-founded tree S there exists a recursive well-founded tree T such that the order-type of T is not smaller than that of S and $PA \vdash \mathrm{Found}(T, \Pi^1_0)$.*

Proof. Fix a recursive non-well-founded tree T' such that PA proves transfinite induction of T' for any arithmetical set, $PA \vdash \mathrm{Found}(T', \Pi^1_0)$, cf. [5] and [6, pp. 280-284]. Let S be some well-founded recursive tree. Let T be the following recursive tree: $\sigma \in T$ iff σ is a sequence of pairs $\langle s_i, t_i \rangle$ $(i < k)$ such that the sequence $\langle s_i : i < k \rangle$ of first components is in S, and the sequence $\langle t_i : i < k \rangle$ of second components is in T'. Then it is easy to see that T is well-founded and that PA proves $\mathrm{Found}(T, \Pi^1_0)$.

Fix an infinite path $\langle t_i : i < \omega \rangle$ through T'. As in [4] p. 437, we can show by induction on $\langle s_i : i < k \rangle \in S$ that

$$|T[\langle \langle s_i, t_i \rangle : i < k \rangle]| \geq |S[\langle s_i : i < k \rangle]|.$$

Hence $|T| \geq |S|$. □

In the following we identify formulas A with the sets defined by them $\{x : A(x)\}$. The formula $\mathrm{Found}(T, A^c)$, where A^c is the complement of A (i.e. $A^c \equiv \neg A$), is equivalent to the following minimization axiom $\mathrm{Min}(T, A)$:

$$A \neq \emptyset \Rightarrow (\exists x \in A)(\forall y \in A)(\neg y \prec_T x).$$

Therefore, the schema $\mathrm{Found}(T, \Pi^0_1)$ is the same as $\mathrm{Min}(T, \Sigma^0_1)$. Σ^0_1 is the class of all r.e. sets. If $\{e\}$ denotes the e-th partial recursive function, or, depending on the context, the e-th recursive enumerable (r.e.) set (we think of $n \in \{e\}$ iff $\{e\}(n) \simeq 0$), then $(\{e\} : e \in \mathbb{N})$ enumerates Σ^0_1. In order to prove Theorem 1 it is enough to show

Lemma 3. *For every prim. rec. well-founded tree S there exists a prim.-rec. well-founded tree T such that the order-type of T is not smaller than that of S and Σ^0_1-IND $\vdash (\forall e) \mathrm{Min}(T, \{e\})$.*

This is obtained by a similar argument as in Lemma 2 from

Theorem 4. *There exists a prim. rec. tree T (actually T will be in $coNP$) which is not well-founded, but which is well-founded for co-r.e. sets, provable in Σ^0_1-IND, i.e. Σ^0_1-IND $\vdash (\forall e) \mathrm{Min}(T, \{e\})$.*

Proof of Lemma 3. We construct T in the same way as in the proof of Lemma 2 from T' given by Theorem 4 and any prim. rec. well-founded tree S. The only additional thing we have to observe is that $(\forall e) \mathrm{Min}(T, \{e\})$ is already provable in Σ^0_1-IND, because the Σ^0_1-sets are uniformly closed under projection. By this we mean that if e is an index of a non-empty set, then we have to consider the 'projection' of $\{e\}$

$$Y = \Big\{ \langle t_1, \ldots, t_k \rangle : (\exists \langle s_1, \ldots, s_k \rangle) \Big(\langle \langle s_1, t_1 \rangle, \ldots, \langle s_k, t_k \rangle \rangle \in \{e\} \Big) \Big\}$$

which again is a Σ^0_1 set, and we can (uniformly in e) find an index for Y. By construction we have Σ^0_1-IND $\vdash \mathrm{Min}(T', Y)$, hence we can find some $t \in Y$ which is $\prec_{T'}$-minimal in Y. Thus, we immediately have some $\tau \in \{e\}$ which is $\prec_T$-minimal in $\{e\}$ □

The idea for the proof of Theorem 4 is as follows. The simplest way to fulfill $\mathrm{Min}(T, \{e\})$ is to ensure that $\{e\} \setminus T \neq \emptyset$ if $\{e\}$ is infinite. For example, we can define an infinite path $P = \{p_j : j \in \omega\}$ (P will also be a tree) by recursion on j, which diagonalizes every infinite r.e. set in the sense that $\{e\} \setminus P \neq \emptyset$ for every infinite set $\{e\}$. Let $e_0 := 0$ (w.l.o.g. $\{0\} = \emptyset$), and for $j \geq 0$ and $p_j := \langle\langle e_1, a_1\rangle, \dots, \langle e_j, a_j\rangle\rangle$ define recursively e_{j+1} as the next index of a set containing some "large" element a_{j+1}:

$$\begin{aligned} e_{j+1} &:= \mu e > e_j.((\exists a > p_j)\ a \in \{e\}) \\ a_{j+1} &:= \mu a > p_j.(a \in \{e_{j+1}\}). \end{aligned}$$

Then we can show

Lemma 5. *For every infinite r.e. set $\{e\}$ we have $\{e\} \setminus P \neq \emptyset$.*

This immediately implies

Theorem 6. *We have $\forall e \in \mathbb{N}\ \mathrm{Min}(P, \{e\})$.* □

Proof of Lemma 5. Let $\{e\}$ be an infinite set. Then there is some i such that $e_i < e \leq e_{i+1}$, because $(e_j)_j$ is strictly increasing. As $\{e\}$ is infinite, we have

$$e > e_i \quad \text{and} \quad (\exists a > p_i)(a \in \{e\}).$$

By definition, e_{i+1} is minimal with this, hence $e_{i+1} \leq e$, hence $e_{i+1} = e$. Now for a_{i+1} we have by definition $a_{i+1} \in \{e\}$ and $p_i < a_{i+1} < p_{i+1}$, hence $a_{i+1} \notin P$. □

Theorem 6 does not imply our desired result, as P is not prim. rec. (rather Π^0_1). We obtain the tree we are looking for by carefully enlarging P to a prim. rec. tree T, whose only infinite path will be similar to P. The idea is to define $\sigma \in T$ by modifying the definition of P by replacing the Σ^0_1-condition "$a \in \{e\}$" by "b witnesses the computation $\{e\}(a) \simeq 0$", thus also replacing "$\langle e_j, a_j\rangle$" by "$\langle e_j, a_j, b_j\rangle$", and by bounding unbounded quantifiers by σ. Of course, this way an arbitrary path $(\sigma_j)_j$ through T with

$$\sigma_j = \langle\langle e_1, a_1, b_1\rangle, \dots, \langle e_j, a_j, b_j\rangle\rangle,$$

which will always fulfill $e_1 < \ldots < e_j$, will usually miss some infinite set $\{e\}$ with $e < e_j$, namely those whose elements $a \in \{e\}$ witnessing their "infiniteness" are greater than σ_j, and those for which verifying $a \in \{e\}$ needs an accepting computation which can not be witnessed below σ_j. But this failure will occur at some point as the path gets longer, and at this point such a "wrong" path will end, and the tree T will branch at some earlier point in this path, such that the "forgotten" index e will now occur. Therefore, all the "wrong" paths will be finite. Another crucial point in constructing T is not hurting the diagonalization property while enlarging P. I.e, for every infinite r.e. set $\{e\}$ we will have some element a in $\{e\} \setminus T$, for which the construction has to ensure that this witness will never be added to the new set.

Let C be the set of accepting configurations, $C := \{\langle e, a, b\rangle : b \text{ witnesses } \{e\}(a) \simeq 0\}$. This set is polytime. With $]a, b[$ we denote the interval $\{a+1, \dots, b-1\}$. For $\sigma = \langle c_1, \dots, c_k\rangle$ and $j \leq k$ let $\sigma \upharpoonright j$ denote the initial subsequence of σ of its first j elements, $\sigma \upharpoonright j = \langle c_1, \dots, c_j\rangle$. Then we define $\sigma \in T$ if and only if

i) $\sigma = \langle\langle e_1, a_1, b_1\rangle, \dots, \langle e_k, a_k, b_k\rangle\rangle \quad \& \quad (\forall\, 0 < j \leq k)\Big(\langle e_j, a_j, b_j\rangle \in C\Big)$

ii) $e_0 := 0 < e_1 < \dots < e_k$

iii) $(\forall j < k)\Big(\sigma \upharpoonright j < a_{j+1}\Big)$

iv) $(\forall j < k)\,(\forall e \in]e_j, e_{j+1}[)\,(\forall a, b < \sigma)\Big(\sigma \upharpoonright j < a \Rightarrow \langle e, a, b\rangle \notin C\Big)$

v) $(\forall j < k)\,(\forall a, b < \sigma)\Big(\sigma \upharpoonright j < a < a_{j+1} \Rightarrow \langle e_{j+1}, a, b\rangle \notin C\Big)$

It is not hard to show that T is a tree and that T contains an infinite path similar to P (to this end extend the definition of P by witnesses b_j for $a_j \in \{e_j\}$). Furthermore, it is immediate from the definition that T is prim. rec. It is possible to change T into a polytime, non-well-founded tree such that Lemma 7 and Proposition 8 still hold, simply blow up $\sigma = \langle c_1, \dots, c_k\rangle$ to $\langle 2^{c_1}, \dots, 2^{c_k}\rangle$ and restrict bounded quantifiers logarithmically, e.g. change $\forall a, b < \sigma \dots$ to $\forall a, b < |\sigma| \dots$. But we will stick to the simpler version as defined before.

With $\mathrm{lh}(\sigma)$ we denote the number of elements in σ, i.e, if $\sigma = \langle s_1, \dots, s_k\rangle$ then $\mathrm{lh}(\sigma) = k$. Let $\prec_{lex}$ be the lexicographic ordering on finite sequences, i.e. if $\sigma = \langle s_1, \dots, s_k\rangle$ and $\tau = \langle t_1, \dots, t_l\rangle$ then let

$$\sigma \prec_{lex} \tau \quad :\Leftrightarrow \quad \sigma \text{ is a proper initial subsequence of } \tau \quad \text{or} \\ \Big(\exists j < \min(k, l)\Big)\Big(\sigma \upharpoonright j = \tau \upharpoonright j \text{ and } s_{j+1} < t_{j+1}\Big).$$

The proof of Proposition 8 is based on the following property of T.

Lemma 7. (Σ^0_0-*IND or* S^1_2) *$\sigma, \tau \in T$ and $\sigma \prec_{lex} \tau \Rightarrow \sigma < \tau$.*

Proof. Let $\sigma, \tau \in T$ such that $\sigma \prec_{lex} \tau$. If σ is a proper initial subsequence of τ we are done. Otherwise define μ to be the greatest common initial subsequence of σ and τ. Let $c = \langle e, a, b\rangle \in C$, $c' = \langle e', a', b'\rangle \in C$ be so that c extends μ in σ and, respectively, c' extends μ in τ. I.e. $\mu^\frown\langle c\rangle$ is an initial subsequence of σ, and $\mu^\frown\langle c'\rangle$ is an initial subsequence of τ. Then $\sigma \prec_{lex} \tau$ and the choice of μ yields $c < c'$. Hence $a, b < c < c' < \tau$.

Assume for the sake of contradiction that $\sigma \geq \tau$. Then we also have $a', b' < c' < \tau \leq \sigma$. Now *iv)* of the definition of $\tau \in T$ and $a, b < \tau$ show $e \geq e'$. Dually, we obtain $e' \geq e$ from *iv)* of $\sigma \in T$ and $a', b' < \sigma$. Hence $e = e'$. Now *v)* of the definition of $\tau \in T$ and $b < \tau$ yields $a \geq a'$, and dually $a' \geq a$ follows from $\sigma \in T$ and $b' < \sigma$, hence $a = a'$. But then also $b = b'$ contradicting $c < c'$. ☐

This lemma immediately implies that there is only one infinite path through T, and that this path is the rightmost one. Furthermore, we can use this lemma to show in Σ^0_1-IND that T is well-founded on co-r.e. sets.

Proposition 8. Σ^0_1-*IND* $\vdash (\forall e)\,\mathrm{Min}(T,\{e\})$.

Proof. We argue in Σ^0_1-IND. Assume for the sake of contradiction that there is some e such that $\neg\,\mathrm{Min}(T,\{e\})$, i.e.

$$\{e\} \neq \emptyset \quad \& \quad (\forall \xi \in \{e\})(\exists \eta \in \{e\})(\eta \prec_T \xi).$$

Then $\{e\} \subset T$, and using Σ^0_1-IND we can define arbitrary long paths in $\{e\} \cap T$ starting from any $\xi \in \{e\}$, i.e. we can show for $\xi \in \{e\}$

$$(\forall k)(\exists \eta)\Big(\eta \in \{e\} \cap T \wedge \eta \prec_T \xi \wedge \mathrm{lh}(\eta) \geq k\Big). \tag{1}$$

In particular, there is a $\sigma \in T \cap \{e\}$ with $\mathrm{lh}(\sigma) > e$, i.e.

$$\sigma = \langle \langle e_1, a_1, b_1\rangle, \ldots, \langle e_k, a_k, b_k\rangle\rangle$$

with $k > e$. By definition $0 =: e_0 < e_1 < \ldots < e_k$, thus $e_k \geq k > e$. Therefore, there must be some $j < k$ such that $e_j < e \leq e_{j+1}$.

If $e_j < e < e_{j+1}$, let a be σ, so $a \in \{e\}$, and b be the witness for $\{e\}(a) \simeq 0$, hence $\langle e, a, b\rangle \in C$. With (1) we obtain some $\tau \in T$ such that $\tau \prec_T \sigma$ and $\mathrm{lh}(\tau) > \max(b, a)$, hence $a, b < \tau$. As $\tau \prec_T \sigma$ we have $\tau \restriction k = \sigma$, hence $\tau \restriction j = \sigma \restriction j < \sigma = a$ contradicting condition *iv*) of $\tau \in T$.

Thus, we must have $e = e_{j+1}$. Then $a_{j+1} \in \{e\} \subset T$ by condition *i*) of $\sigma \in T$, and condition *iii*) yields

$$\sigma \restriction j < a_{j+1} < \langle e_{j+1}, a_{j+1}, b_{j+1}\rangle < \sigma \restriction (j+1),$$

hence a_{j+1} is incomparable to σ according to the initial subsequence relation. Now either $a_{j+1} \prec_{lex} \sigma$ or $\sigma \prec_{lex} a_{j+1}$. As $a_{j+1} < \sigma$ and $a_{j+1}, \sigma \in T$, we obtain $\sigma \not\prec_{lex} a_{j+1}$ by Lemma 7. Hence $a_{j+1} \prec_{lex} \sigma$. With (1) we find some $\tau \in T$ with $\tau \prec_T a_{j+1}$ and $\mathrm{lh}(\tau) \geq \sigma$, thus $\tau \geq \sigma$. As a_{j+1} is incomparable to σ, and $a_{j+1} \prec_{lex} \sigma$, we obtain $\tau \prec_{lex} \sigma$ from $\tau \prec_T a_{j+1}$, contradicting Lemma 7. □

In the proof of the last theorem induction is only needed to obtain (1) from $\neg\,\mathrm{Min}(T,\{e\})$. We can hide the induction by changing the definition of $\mathrm{Min}(T,A)$ to $\widetilde{\mathrm{Min}}(T,A)$ of the form

$$A \neq \emptyset \rightarrow (\exists x \in A)(\exists k)(\forall y \in A)(\mathrm{lh}(y) \geq k \rightarrow \neg y \prec_T x)$$

i.e. $(\forall e)\widetilde{\mathrm{Min}}(T,\{e\})$ is provable without induction (e.g. in Σ^0_0-IND or S^1_2). Furthermore, $\mathrm{Min}(T,A)$ and $\widetilde{\mathrm{Min}}(T,A)$ are equivalent in the standard model.

We finish the paper with some remarks and questions.

1. Let $\{e\}^{\Sigma^0_n}$ denote the e-th partial recursive function in some fixed Σ^0_n-complete oracle (e.g., the n-th Touring jump). Then $\{e\}^{\Sigma^0_n}$ enumerates all Σ^0_{n+1}-sets. If we replace $\{e\}$ in the definition of C by $\{e\}^{\Sigma^0_n}$ obtaining C_n and T_n in the following, then $C_n \in \Delta^0_0(\Sigma^0_n)$ and hence $T_n \in \Delta^0_0(\Sigma^0_n)$. Thus T_n is a Δ^0_{n+1}-tree which is not well-founded but which is well-founded for all Π^0_{n+1}-sets, provable in Σ^0_{n+1}-IND.

2. Most of our arguments are constructive as far as the proof of Proposition 8, even if we think of proving $\forall e\,\mathrm{Found}(T,\{e\}^c)$. In order to prove $\forall e\,\mathrm{Found}(T,\{e\}^c)$ constructively, we would have to derive contradiction from the assumptions $(\forall x)((\forall y \prec x)(y \notin \{e\}) \to x \notin \{e\})$ and $(\exists x)x \in \{e\}$ for some arbitrary e. But we do not obtain arbitrary long descending paths in $\{e\} \cap T$ (cf. (1)) constructively from these assumptions. So the question is: Does there exist a prim. rec. relation $\prec$ such that $\mathrm{Found}(\prec,\Pi^0_1)$ is provable in some intuitionistic theory like Σ^0_1-ind or HA? In [3] it is shown that there exists a Π^0_2-formula $A(a)$ such that if $HA \vdash \mathrm{Found}(\prec,A)$ holds for prim. rec. $\prec$ then $\prec$ is well-founded. Hence, even if we find such a $\prec$ which answers our question, we cannot expect to extend the result to higher levels.

3. We directly have $\mathrm{O}_{\Sigma^0_n}(\Sigma^0_1\text{-IND}) \geq \mathrm{O}_{\Pi^0_{n+1}}(\Sigma^0_1\text{-IND})$ and $\mathrm{O}_{\Pi^0_n}(\Sigma^0_1\text{-IND}) \geq \mathrm{O}_{\Pi^0_{n+1}}(\Sigma^0_1\text{-IND})$, because $\Sigma^0_n \cup \Pi^0_n \subset \Pi^0_{n+1}$. We do not know what the values of $\mathrm{O}_{\Pi^0_2}(\Sigma^0_1\text{-IND})$, $\mathrm{O}_{\Pi^0_3}(\Sigma^0_1\text{-IND})$, $\dots$, or $\mathrm{O}_{\Pi^1_0}(\Sigma^0_1\text{-IND})$ are. I.e., is $\mathrm{O}_{\Pi^1_0}(\Sigma^0_1\text{-IND}) < \omega^{CK}_1$? Which is the first i (if any) such that $\mathrm{O}_{\Pi^0_i}(\Sigma^0_1\text{-IND}) < \omega^{CK}_1$? What happens if we consider other suitable formalizations of Min like $\widetilde{\mathrm{Min}}$? E.g., does there exist a non-well-founded prim. rec. tree T' such that $\Sigma^0_0\text{-IND} \vdash \widetilde{\mathrm{Min}}(T',\Pi^1_0)$?

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