Linear Kripke Frames and Gödel Logics*

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August 11, 2006

Abstract

We investigate the relation between intermediate predicate logics based on countable linear Kripke frames with constant domains and Gödel logics. We show that for any such Kripke frame there is a Gödel logic which coincides with the logic defined by this Kripke frame on constant domains and vice versa. This allows us to transfer several recent results on Gödel logics to logics based on countable linear Kripke frames with constant domains: We obtain a complete characterisation of axiomatisability of logics based on countable linear Kripke frames with constant domains.[1] Furthermore, we obtain that the total number of logics defined by countable linear Kripke frames on constant domains is countable.

1 Introduction

Kripke frames as possible semantics for modal logics were introduced by S. A. Kripke in the late fifties and early sixties. While the origin of this notion is disputable, the influence of the *possible world interpretation* has been enormous. This new type of semantics provided an attractive model theory that seemed more manageable than the previous algebra-based semantics. One of the reasons for its early success was that well known logical systems, like S4, S5 and Intuitionistic Predicate Logic, were shown to be characterised by natural first-order properties of their frames. Kripke himself in [Kri65] used these frames to prove the completeness of Intuitionistic Predicate Logic. For a more detailed presentation of these and related topics see [Gab81, Gol03].

Detailed studies of intermediate predicate logics based on linear Kripke frames have been carried out by several researchers. For example, the general structure of linear Kripke frames and their logics is discussed in [Ono88], the logics defined by Kripke frames determined by ordinals on constant domains are analysed in [MTO90], and the logics based on Kripke frames $\mathbb{R}$ and $\mathbb{Q}$ with constant domains are determined in [Tak87b].

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*The results of this paper have been presented at the ESF Exploratory Workshop "The Challenge of Semantics" in Vienna, July 2004.

†Supported in part by FWF-grants #P17503-MATH and #P16539-N04 of the Austrian Science Fund.

‡Supported by the European Union under EC-MC 008054 and FWF-grant #P16539-N04 of the Austrian Science Fund.

[1] Skvortsov [Skv05] recently has announced similar results on the characterisation of axiomatisability.

Propositional finite-valued Gödel logics were introduced in 1933 by Gödel [Göd33] to show that Propositional Intuitionistic Logic does not have a finite characteristic matrix, i.e. that Propositional Intuitionistic Logic is not a finite-valued logic. Dummett in [Dum59] was the first to study infinite valued Gödel logics, axiomatising the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the axiom scheme $(A \to B) \vee (B \to A)$ of linearity. He showed that there is only one infinite valued propositional Gödel logic. Infinite-valued propositional Gödel logic is also called Gödel-Dummett logic or Dummett's LC.

Standard first-order Gödel logics – the one based on the full $[0, 1]$ interval as the truth value set – have been studied under several names in the last decades (see Section 3 for the definition of Gödel logics). Horn in [Hor69] was probably the first to refer to this logic by the name *logic with truth values in a linearly ordered Heyting algebra*. He also gave the first completeness proof. Later on Takeuti and Titani introduced *intuitionistic fuzzy logic* in [TT84], and also proved the completeness of their system. This system incorporates the density rule

$$\frac{\Gamma \vdash A \vee (C \to p) \vee (p \to B)}{\Gamma \vdash A \vee (C \to B)}$$

(where p is any propositional variable not occurring in the lower sequent.) This rule exhibits an interesting property: It forces the truth value set to be dense in itself. This cannot be achieved by formulas, exhibiting the difference in expressive power of rules versus formulas in Gödel logics. Finally Takano [Tak87a] has shown that the system given by Horn and the one given by Takeuti/Titani are equivalent and strongly complete. Furthermore, he gave a semantic elimination of the density rule. A syntactical proof of the elimination of the density rule was later given by Baaz and Zach in [BZ00].

The study of general first-order Gödel logics, i.e. those based on truth value sets different from the unit interval, has been initiated by Matthias Baaz et al. in the mid-nineties (cf. [BLZ96a, BLZ96b, Baa96, BZ98]). One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values (cf. [BLZ96b, Baa96, Pre02]). In the light of a general result of Scarpellini in [Sca62] on non-axiomatisability of infinite-valued first-order Łukasiewicz logic which can be extended to almost all linearly ordered infinite-valued logics, it is surprising that some of the infinite-valued Gödel logics are among the few recursively enumerable linearly ordered first-order logics (cf. [Hor69, TT84, Pre03, BPZ]). Recently, Gödel logics have received increasing attention, both in terms of foundational investigations and in terms of applications, as they have been recognised as one of the most important formalisations of fuzzy logic and one of the three fundamental t-norm based logics (cf. [Háj98]).

Due to the genesis of Gödel logics as described above we can expect that there is a strong relation between logics defined via Kripke frames and Gödel logics. Beside the already mentioned linearity axiom scheme, the second important axiom scheme valid in all Gödel logics is the quantifier shift $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$, where x must not occur in A. For logics defined by Kripke frames, these two properties of linearity and quantifier shift induce the linearity of the accessibility relation and the condition on constant domains.

For propositional logic the truth value sets on which Gödel logics are based can be considered as linear Heyting algebras (or pseudo-Boolean algebras). By taking the prime filters of a Heyting algebra as the Kripke frame it is easy to see that the induced logics coincide (cf. [Fit69, Ono71]). This direct method does not work for first order logics as the structure of the prime filters does not coincide with the possible evaluations in the first order case, see Remark on page 6.

The present paper shows that the class of logics defined by countable linear Kripke frames on constant domains and the class of all Gödel logics coincide. More precisely, for every countable Kripke frame we will *construct* a truth value set such that the logic induced by the Kripke frame and the one induced by the truth value set coincide, and vice versa (Theorems 18 and 25). As corollaries we obtain a complete characterisation of axiomatisability of logics based on countable linear Kripke frames with constant domains (Corollaries 27 and 28). Furthermore, we obtain that there are only countable many different logics based on countable linear Kripke frames with constant domains (Corollary 30). This is especially surprising for at least two reasons: Due to a result obtained in [BZ98] there are uncountably many different propositional quantified Gödel logics, and thus also uncountably many propositional quantified logics based on countable linear Kripke frames. Furthermore, the number of all intermediate (predicate) logics extending the basic linear logic with constant domains is uncountable.

2 Algebraic preliminaries

The following section discusses algebraic properties of partial orders and lattices which are relevant for the results in the present article. All definitions, lemmas and propositions are well known from classical lattice theory, and we will cite relevant articles and textbooks as references for more detailed treatments. For a general treatment of lattice theory we refer the reader to [Bir67].

Definition 1 (Partial order). A *partial order* is a pair $(P, \preceq)$ consisting of a set P and a reflexive, transitive and antisymmetric binary relation $\preceq$ on P.

Definition 2 (Upsets). Let $\mathcal{P} = (P, \preceq)$ be a partial order. A subset $X \subseteq P$ is called *upward closed w.r.t.* $\preceq$ iff for all x, x', if $x \in X$ and $x \preceq x'$, then $x' \in X$. Upward closed sets will also be called *upsets*. With $\mathrm{Up}(\mathcal{P})$ we denote the set of all upsets of $\mathcal{P}$.

We often identify $\mathrm{Up}(\mathcal{P})$ with the partial order $(\mathrm{Up}(\mathcal{P}), \subseteq)$. $\mathcal{P}$ is naturally embedded into $\mathrm{Up}(\mathcal{P})$ by mapping an element $p \in P$ to the upset $p^{\uparrow} := \{p' \in P \colon p \preceq p'\}$.

Observe that the set of upsets $\mathrm{Up}(\mathcal{P})$ forms a complete and completely distributive lattice.

Definition 3 (Complete ring of sets [Ran52]). A family $\mathscr{S}$ of subsets of a set is called a *complete ring of sets* if for every $U \subset \mathscr{S}$ the intersection $\bigcap U$ and union $\bigcup U$ are in $\mathscr{S}$. A complete ring of sets is called *linear* if it is linearly ordered by $\subseteq$. Given a complete linear ring of sets we define $\mathbf{1} = \bigcup \mathscr{S}$ and $\mathbf{0} = \bigcap \mathscr{S}$.

Note that a complete linear ring of sets forms a complete lattice, and hence a complete Heyting algebra (cf. [Ran52]). The residuum $\to$ of $\cap$ is given by

$$s \to t = \begin{cases} \mathbf{1} & \text{if } s \subseteq t \\ t & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two algebraic structures. A mapping is called an $\mathcal{A}_1$–$\mathcal{A}_2$ *homomorphism* if it maps the objects of $\mathcal{A}_1$ to the objects of $\mathcal{A}_2$, and honours the respective operations. An injective, surjective, or bijective homomorphism is called *monomorphism*, *epimorphism*, or *isomorphism*, respectively. For example, let $\mathscr{S}$ be a complete ring of sets, and $(B, \leq, \bigwedge, \bigvee)$ be a complete Heyting algebra. Then σ is a $(\mathscr{S}, \subseteq, \bigcap, \bigcup)$–$(B, \leq, \bigwedge, \bigvee)$ homomorphism, if σ is a mapping from $\mathscr{S}$ to B which is monotone (w.r.t. $\subseteq$–$\leq$) and respects the other operations, i.e. for $X \subseteq \mathscr{S}$ we have $\sigma(\bigcap X) = \bigwedge \{\sigma(x) \colon x \in X\}$ and $\sigma(\bigcup X) = \bigvee \{\sigma(x) \colon x \in X\}$.

It is well-known that any complete ring of sets is isomorphic to the lattice of all order ideals of some partial order and vice versa (e.g., see [Dav79].) In our somewhat simpler situation of complete linear rings of sets versus total orders we obtain the following two observations. Both observations have easy direct proofs. We will state the proof of the second one for convenience of the reader.

Proposition 4. *The upsets of a partial order form a complete ring of sets. The upsets of a total order form a complete linear ring of sets.*

Proposition 5. *For every complete (linear) ring of sets $\mathscr{S}$ there is a (linear) partial order $\mathcal{P}$ such that* $\mathrm{Up}(\mathcal{P})$ *is isomorphic to $\mathscr{S}$.*

Proof. Define $P' = \bigcup \mathscr{S}$ and for p, p' in P' define $p \preceq' p'$ iff for all $s \in \mathscr{S}$ with $p \in s$, $p' \in s$ also holds. The relation $p \sim p'$ iff $p \preceq' p' \wedge p' \preceq' p$ is an equivalence relation on P'. Define $P = P'/\!\sim$ and a binary relation $\preceq$ on P by $[p]_\sim \preceq [p']_\sim$ iff $p \preceq' p'$. Let $\mathcal{P} = (P, \preceq)$, then $\mathrm{Up}(\mathcal{P})$ is isomorphic to $\mathscr{S}$. □

3 Intermediate predicate logics on constant domains

Let $\mathcal{L}$ be a countable first-order language which includes the propositional constant $\bot$. For a set U, let $\mathcal{L}^U$ denote the extended language with constants for all elements of U. The set of all closed atomic formulas of $\mathcal{L}^U$ is denoted by $\mathbb{A}(\mathcal{L}^U)$, the set of all sentences of $\mathcal{L}^U$ is denoted by $\mathbb{S}(\mathcal{L}^U)$.

A *Kripke frame* is a partial order $K = (W, \preceq)$. We are interested in the relation between Gödel logics and logics defined by Kripke frames. As we have mentioned in the introduction, the validity of the axiom scheme $(A \to B) \vee (B \to A)$ of linearity in all Gödel logics transfers to linear Kripke frames and the validity of the quantifier shift $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$ where x must not occur in A transfers to the property of constant domains. Therefore, in the following we will only consider linear Kripke frames, i.e. total orders, and we will denote the order by $\preceq$. Furthermore, we will only consider logics defined by linear Kripke frames on constant domains. Elements of Kripke frames will also be called *worlds*.

With the following definitions we adapt the standard definitions (which can be found in [Ono73] for example) to the special case of Kripke models based on linear frames with constant domains.

Definition 6. Let $K = (W, \preceq)$ be a linear Kripke frame. For any non-empty set U, the tuple (K, U) is called a *Kripke model*. A relation $\mathrm{val}_{(K,U)} \subseteq W \times \mathbb{A}(\mathcal{L}^U)$ is said to be a *valuation* of the Kripke model (K, U) if $\mathrm{val}_{(K,U)}$ satisfies the following conditions: For all w, w_1 and w_2 in W it holds that $w_1 \preceq w_2$ and $\mathrm{val}_{(K,U)}(w_1, \varphi)$ implies $\mathrm{val}_{(K,U)}(w_2, \varphi)$, and that $\mathrm{val}_{(K,U)}(w, \bot)$ does not hold.

The valuation $\mathrm{val}_{(K,U)}$ can be extended to a relation on $W \times \mathbb{S}(\mathcal{L}^U)$ inductively:

$$
\begin{array}{rcl}
\mathrm{val}_{(K,U)}(w, \varphi \wedge \psi) & \text{iff} & \mathrm{val}_{(K,U)}(w, \varphi) \text{ and } \mathrm{val}_{(K,U)}(w, \psi) \\
\mathrm{val}_{(K,U)}(w, \varphi \vee \psi) & \text{iff} & \mathrm{val}_{(K,U)}(w, \varphi) \text{ or } \mathrm{val}_{(K,U)}(w, \psi) \\
\mathrm{val}_{(K,U)}(w, \varphi \to \psi) & \text{iff} & \mathrm{val}_{(K,U)}(v, \varphi) \text{ implies } \mathrm{val}_{(K,U)}(v, \psi) \\
 & & \text{for any } v \text{ with } w \preceq v \\
\mathrm{val}_{(K,U)}(w, \forall x \varphi(x)) & \text{iff} & \mathrm{val}_{(K,U)}(w, \varphi(u)) \text{ for any } u \in U \\
\mathrm{val}_{(K,U)}(w, \exists x \varphi(x)) & \text{iff} & \mathrm{val}_{(K,U)}(w, \varphi(u)) \text{ for some } u \in U \ .
\end{array}
$$

Definition 7. The *logic defined by a linear Kripke frame $K = (W, \preceq)$ on constant domains*, denoted by $\mathbf{L}(K)$, is the set of all $\mathcal{L}$-formulas φ such that for all Kripke models (K, U), all valuations $\mathrm{val}_{(K,U)}$ of (K, U), and all worlds $w \in W$, $\mathrm{val}_{(K,U)}(w, \varphi')$ holds, where φ' is a closure of φ.

Logics of linearly ordered Kripke frames in general have been studied, e.g., by Ono [Ono88]. Some special cases of logics defined by linearly ordered Kripke frames on constant domains have also been considered in the literature: for example, $\mathbf{L}(\mathbb{Q})$ and $\mathbf{L}(\mathbb{R})$ in [Tak87b], and $\mathbf{L}(\alpha)$ for ordinals α in [MTO90].

We continue by defining logics based on complete linear rings of sets, which will be used as a turning point between logics of Kripke frames and Gödel logics. They form a special case of logics defined by complete pseudo-Boolean algebras (also called complete Heyting algebras) (cf. [Ono73]).

Definition 8. Let $\mathscr{S}$ be a complete linear ring of sets. Recall that $\mathbf{1} = \bigcup \mathscr{S}$ and $\mathbf{0} = \bigcap \mathscr{S}$. A tuple $(\mathscr{S}, U)$ is called a *model based on* $\mathscr{S}$ *and* U if U is a non-empty set. An *assignment* $\mathrm{val}_{(\mathscr{S},U)}$ *of* $(\mathscr{S}, U)$ is a mapping from $\mathbb{A}(\mathcal{L}^U)$ to $\mathscr{S}$ with $\mathrm{val}_{(\mathscr{S},U)}(\bot) = \mathbf{0}$.

The assignment $\mathrm{val}_{(\mathscr{S},U)}$ can be extended to a function from $\mathbb{S}(\mathcal{L}^U)$ to $\mathscr{S}$ inductively:

$$\begin{aligned}
\mathrm{val}_{(\mathscr{S},U)}(\varphi \wedge \psi) &= \mathrm{val}_{(\mathscr{S},U)}(\varphi) \cap \mathrm{val}_{(\mathscr{S},U)}(\psi) \\
\mathrm{val}_{(\mathscr{S},U)}(\varphi \vee \psi) &= \mathrm{val}_{(\mathscr{S},U)}(\varphi) \cup \mathrm{val}_{(\mathscr{S},U)}(\psi) \\
\mathrm{val}_{(\mathscr{S},U)}(\varphi \to \psi) &= \begin{cases} \mathbf{1} & \text{if } \mathrm{val}_{(\mathscr{S},U)}(\psi) \subseteq \mathrm{val}_{(\mathscr{S},U)}(\varphi) \\ \mathrm{val}_{(\mathscr{S},U)}(\psi) & \text{otherwise} \end{cases} \\
\mathrm{val}_{(\mathscr{S},U)}(\forall x \varphi(x)) &= \bigcap \left\{ \mathrm{val}_{(\mathscr{S},U)}(\varphi(u)) \colon u \in U \right\} \\
\mathrm{val}_{(\mathscr{S},U)}(\exists x \varphi(x)) &= \bigcup \left\{ \mathrm{val}_{(\mathscr{S},U)}(\varphi(u)) \colon u \in U \right\} \quad .
\end{aligned}$$

Following [Ono73] we define the logic based on a complete linear ring of sets.

Definition 9. The *logic defined by a complete linear ring of sets* $\mathscr{S}$, denoted by $\mathbf{L}(\mathscr{S})$, is the set of $\mathcal{L}$-formulas φ such that for all models based on $\mathscr{S}$ and some non-empty set U and all assignments $\mathrm{val}_{(\mathscr{S},U)}$, $\mathrm{val}_{(\mathscr{S},U)}(\varphi') = \mathbf{1}$ holds, where φ' is a closure of φ.

The equivalence of Kripke models based on valuations and Kripke models based on corresponding upsets (or 'propositions') is well known (cf. [Gab81, Chapter IX] and [Kre97]). Using this equivalence we obtain the following Lemma.

Lemma 10. *Let* $K = (W, \preceq)$ *be a linear Kripke frame and* $\mathrm{Up}(K)$ *the induced complete linear ring of sets. Then* $\mathbf{L}(K) = \mathbf{L}(\mathrm{Up}(K))$.

Proof. For atomic formulas, the natural correspondence between valuations based on K and U and assignments based on $\mathrm{Up}(K)$ and U is given by

$$\mathrm{val}_{(\mathrm{Up}(K),U)}(\varphi) = \{w \colon \mathrm{val}_{(K,U)}(w, \varphi)\}.$$

This equivalence immediately extends to all formulas by induction. □

As mentioned in the introduction, the study of first-order Gödel logics based on general truth value sets (which we will call 'Gödel sets' for convenience) has been initiated by Baaz et.al. in [BLZ96a, BLZ96b, Baa96, BZ98].

Definition 11 (Gödel set). A *Gödel set* is a set $V \subseteq [0, 1]$ which is closed and contains 0 and 1.

Definition 12 (Semantics of Gödel logic [BLZ96b]). Fix a Gödel set V and a non-empty set U. An *interpretation* $\mathscr{I}$ *based on* V *and* U is a mapping from $\mathbb{A}(\mathcal{L}^U)$

to V such that $\mathscr{I}(\bot) = 0$. Given an interpretation $\mathscr{I}$, we can naturally define a value $\mathscr{I}(\varphi)$ for any sentence $\varphi \in \mathbb{S}(\mathcal{L}^U)$ by induction:

$$\begin{aligned}
\mathscr{I}(\varphi \wedge \psi) &= \min(\mathscr{I}(\varphi), \mathscr{I}(\psi)) \\
\mathscr{I}(\varphi \vee \psi) &= \max(\mathscr{I}(\varphi), \mathscr{I}(\psi)) \\
\mathscr{I}(\varphi \to \psi) &= \begin{cases} 1 & \text{if } \mathscr{I}(\varphi) \leq \mathscr{I}(\psi) \\ \mathscr{I}(\psi) & \text{otherwise} \end{cases} \\
\mathscr{I}(\forall x \varphi(x)) &= \inf\{\mathscr{I}(\varphi(u)) \colon u \in U\} \\
\mathscr{I}(\exists x \varphi(x)) &= \sup\{\mathscr{I}(\varphi(u)) \colon u \in U\} \ .
\end{aligned}$$

For this definition it is essential that Gödel sets V are *closed* subsets of $[0,1]$, so that the interpretation of $\forall$ and $\exists$ through inf and sup again produce values in V.

Let V be a Gödel set. For any $\mathcal{L}$-sentence φ and any Gödel set V let

$$\|\varphi\|_V := \inf\{\mathscr{I}(\varphi) \colon \mathscr{I} \text{ is an interpretation based on } V \text{ and some non-empty set } U\} \ .$$

For an $\mathcal{L}$-formula φ we define $\|\varphi\|_V := \|\varphi'\|_V$ where φ' is a closure of φ.

Definition 13 (Gödel logics based on V [BLZ96b]). For a Gödel set V we define the *first order Gödel logic* $\mathbf{G}_V$ as the set of all $\mathcal{L}$-formulas φ such that $\|\varphi\|_V = 1$. If $\varphi \in \mathbf{G}_V$ we also write $\mathbf{G}_V \models \varphi$.

Let $\mathscr{S}$ be a complete linear ring of sets and V be a Gödel set. A direct observation is that if there is a $(\mathscr{S}, \subseteq, \bigcap, \bigcup)$–$(V, \leq, \inf, \sup)$ isomorphism then $\mathbf{L}(\mathscr{S}) = \mathbf{G}_V$. As $\mathscr{S}$ is complete and V is closed this is already induced from $(\mathscr{S}, \subseteq)$ and $(V, \leq)$ being isomorphic. Therefore, we have the following Proposition:

Proposition 14. *Let $\mathscr{S}$ be a complete linear ring of sets and V be a Gödel set such that there is a $(\mathscr{S}, \subseteq)$–$(V, \leq)$ isomorphism. Then $\mathbf{L}(\mathscr{S}) = \mathbf{G}_V$.*

Remark. For propositional logic the truth value sets on which Gödel logics are based can be considered as linear Heyting algebras. By taking the prime filters of a Heyting algebra as the Kripke frame one can see that the induced logics coincide (cf. [Fit69, Ono71]). This direct method does not work for predicate logics as the structure of the prime filters does not coincide with the possible evaluations in the first order case. For example, using the order-theoretic notions ω for $(\mathbb{N}, <)$ and ω^* for $(\mathbb{N}, >)$, let us consider the Kripke frame $K = \omega + \omega^*$. The order structure of the upsets of K can be described as $\omega + 1 + \omega^*$. The upsets of K form a complete Heyting algebra, whose prime filters have an order structure isomorphic to $\omega + 1 + 1 + \omega^*$. The intermediate predicate logics defined by the Kripke frame K on constant domains and by the Heyting algebra $\mathrm{Up}(K)$ are the same, but different from the intermediate predicate logic defined by the Kripke frame $\omega + 1 + 1 + \omega^*$ on constant domains, as the following example shows.

Example 15. Let $K_1 = \omega + \omega^*$, $K_2 = \omega + 1 + \omega^*$, and $K_3 = \omega + 1 + 1 + \omega^*$. Then the intermediate predicate logics defined by K_1 and K_2 are incomparable, and the intermediate predicate logic defined by K_3 is a proper subset of the one defined by K_2. To see this, let $\varphi < \psi$ denote the formula $(\psi \to \varphi) \to \psi$. Let P and Q be unary predicate symbols, and R be a predicate constant. We can pin down the parts ω and ω^* in the frames using the formulas $\varphi_P = \forall x(\forall y Py < Px)$ and $\varphi_Q = \forall x(Qx < \exists y Qy)$, respectively, in the sense that under a given valuation $\mathrm{val}_{(K,U)}$ the former formula evaluates to true iff the set $\{\mathrm{val}_{(\mathrm{Up}(K),U)}(Pu) \colon u \in U\}$ corresponds to ω, and the latter formula evaluates to true iff the set $\left\{\mathrm{val}_{(\mathrm{Up}(K),U)}(Qu)\colon u \in U\right\}$ corresponds to ω^*. Thus we obtain:

$$\begin{aligned} \varphi_P \wedge \varphi_Q \to (\forall y Py \to \exists x Qx) &\in \mathbf{L}(K_1) \setminus \mathbf{L}(K_2) \\ \varphi_P \wedge \varphi_Q \to (\exists x Qx < \forall y Py) &\in \mathbf{L}(K_2) \setminus \mathbf{L}(K_1) \\ \varphi_P \wedge \varphi_Q \wedge (\exists x Qx < R) \to (\forall y Py \to R) &\in \mathbf{L}(K_2) \setminus \mathbf{L}(K_3) \ . \end{aligned}$$

The topic of the present paper indicates a need of understanding when the logics induced by two semantical structures are the same. We close this subsection with a short discussion of notions for the different semantics that imply inclusion of the induced logics. Considering Kripke frames as partial orders one would expect that the existence of a $\leq$-epimorphism between two Kripke frames results in the logics being ordered by inclusion. But it turns out that being a $\leq$-epimorphism does not necessarily mean that the valuation properties are respected. The adequate concept is the one of being a p-morphism: a $\leq$-epimorphism is a p-morphism if is satisfies the additional property that whenever $f(x) = y$ and $y \leq y'$, then there is a $x' \geq x$ such that $f(x') = y'$ (cf. [Ono73, CZ97]).

Considering complete rings of sets, which are complete Heyting algebras, the adequate concept for implying inclusion of the induced logics are complete monomorphisms, which are those injections which respect the operations of the Heyting algebra and which in addition preserve arbitrary unions and intersections. The latter is equivalently expressed by saying that they are monomorphisms of the complete ring of sets, as the general intersection and union are operations of these algebras.

The relation between these structures can be seen in categorical terms: The category of Kripke frames with p-morphism is dual to the category of complete Heyting algebras where all elements are joins of completely join-irreducible elements, together with complete monomorphism (cf. [Esa85]). The linear case opens a much simpler perspective to these relations as the concept of p-morphisms collapses to surjections. Hence, if $f : K \to K'$ is an epimorphism, then there exists a monomorphism $\bar{f} : \mathrm{Up}(K') \to \mathrm{Up}(K)$ such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\ f\ } & K' \\ {\scriptstyle w^{\uparrow}}\big\downarrow & & \big\downarrow{\scriptstyle w^{\uparrow}} \\ \mathrm{Up}(K) & \xleftarrow[\ \bar{f}\ ]{} & \mathrm{Up}(K') \end{array}$$

Furthermore, the converse is also true: For every monomorphism $\bar{f} : \mathrm{Up}(K') \to \mathrm{Up}(K)$ there exists an epimorphism $f : K \to K'$ such that the above diagram commutes. Observe that in both these cases $\mathbf{L}(K)$ $(= \mathbf{L}(\mathrm{Up}(K)))$ is included in $\mathbf{L}(K')$ $(= \mathbf{L}(\mathrm{Up}(K')))$.

3.1 From Kripke frames to Gödel sets

Going from Kripke frames to Gödel logics we have to construct for any Kripke frame K a Gödel set V_K such that the logic induced by K and the Gödel logic defined by V_K coincide. The first idea – take the order structure of K, embed it into $\mathbb{R}$ and then take the topological closure – seems to be the trivial solution, and in fact it works e.g. for the Kripke frame of type ω^*, i.e. the inverse order of ω. Embedding this frame into $[0,1]$ and taking its closure we obtain a truth value set isomorphic to $V_{\uparrow} = \{1 - 1/n\colon n \geq 1\} \cup \{1\}$. But as soon as we consider more complex Kripke frames we run into trouble. For example, consider the Kripke frame described by $\mathbf{1} + \omega^*$, that is one initial element followed by an inverse order of ω. The naive translation would result in the same Gödel set $V_{\uparrow}$, but these logics do not coincide, because the sentence $\exists x(\exists y P(y) \to P(x))$ (expressing in the language of Gödel logics that there are no proper suprema in the truth value set besides 1) is valid in $V_\uparrow$ but not in the logic defined by $\mathbf{1}+\omega^*$. The reason why this naive method fails is obvious as soon as one considers on the Kripke side the possible 'valuations' of atomic formulas, i.e., the sets of worlds in which atomic formulas can be valid. This is exactly the collection of upsets on K, $\mathrm{Up}(K)$. Hence, in fact what has to be transfered into $[0,1]$ is not the actual Kripke worlds, but the order structure of $\mathrm{Up}(K)$ in such a way that the order theoretic infima and suprema are transfered into topological infima and suprema.

Now we also see that the problematic worlds in the original Kripke frame are those worlds which are 'proper' order theoretic infima, i.e. those worlds w where for any w' such that $w \prec w'$ there is a w'' such that $w \prec w'' \prec w'$. Such worlds will be called *limit worlds*. In case of the example above of a Kripke frame with order $\mathbf{1}+\omega^*$ we see that the first world is such a limit world and that the set of upward closed subsets of W contains another element, let's call it $\mathbf{1}^*$, which contains all the elements from ω^* but not the element $\mathbf{1}$ itself. Hence, the correct Gödel set modulo isomorphisms is $V'_\uparrow = \{0.9 - 1/n \colon n \geq 2\} \cup \{0, 0.9, 1\}$. And in fact, in the logic $\mathbf{G}_{V'_\uparrow}$, the sentence $\exists x(\exists y P(y) \to P(x))$ is not valid. Moreover, we will see in the following that the logics $\mathbf{G}_{V'_\uparrow}$ and $\mathbf{L}(\mathbf{1}+\omega^*)$ are the same indeed.

For the construction of a Gödel set from a Kripke frame $K = (W, \preceq)$ we first study the different types of upsets which may occur. For $s \in \mathrm{Up}(K)$ let s^c be the complement of s w.r.t. K, i.e. $s^c := W \backslash s$. There are four types of upsets $s \in \mathrm{Up}(K)$, depending on whether or not s^c contains a maximal element and whether or not s contains a minimal element. For our further discussions it will be convenient to group two of them into one type which we denote α. The other two we shall call β and γ. In order to omit tedious case distinction we define $\bigcap \emptyset := W$, $\bigcup \emptyset := \emptyset$, $\inf \emptyset := 1$, and $\sup \emptyset := 0$.

For $s \in \mathrm{Up}(K)$ we define the *type of* s, $\mathrm{tp}(s) \in \{\alpha, \beta, \gamma\}$, as follows:

$\mathrm{tp}(s) = \alpha$	iff	s contains a minimal element.
$\mathrm{tp}(s) = \beta$	iff	s is not of type α, but s^c contains a maximal element.
$\mathrm{tp}(s) = \gamma$	iff	s is not of type α and s is not of type β.

If s is of type γ, it is easy to see that $s = \bigcap \{w^\uparrow \colon w \in s^c\}$. Fig. 1 shows the three different types of upsets together with their images in the Gödel set. The upsets s_1, s_2, s_3, and s_4 are of types α, β, α, and γ, respectively.

We recall the following lemma from [Hor69]:

Lemma 16 ([Hor69, Lemma 3.7]). *Let $\mathcal{P} = (P, \preceq)$ be a countable linear order, then there exists a monomorphism from $\mathcal{P}$ to $(\mathbb{Q} \cap [0,1], \leq)$ which preserves infima and suprema already existing in $\mathcal{P}$.*

Since we have to extend this lemma we present the construction of the monomorphism, but leave the verification of the properties to the reader.

Proof of Lemma 16. If $\mathcal{P}$ does not have a first or last element, we add the missing ones and denote the resulting linear order again by $\mathcal{P}$. Then denote the first and the last element of $\mathcal{P}$ by $\mathbf{0}$ and $\mathbf{1}$, respectively. Now assume that the members of P are enumerated in the form: $a_0 = \mathbf{0}$, $a_1 = \mathbf{1}$, $a_2, \ldots$. To avoid misunderstandings, we point out that $\mathbf{0} \prec a_i \prec \mathbf{1}$ for all $i \geq 2$. Let $h(\mathbf{0}) = 0$ and $h(\mathbf{1}) = 1$ and define $h(a_n)$ inductively: Let $a_n^- := \max\{a_i \colon i < n \text{ and } a_i \prec a_n\}$ and $a_n^+ := \min\{a_i \colon i < n \text{ and } a_i \succ a_n\}$, and define

$$h(a_n) := \frac{h(a_n^-) + h(a_n^+)}{2} \quad .$$

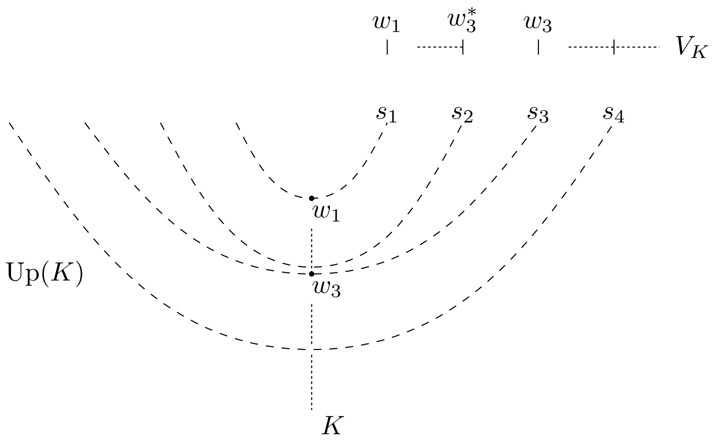

Figure 1: Different types of upsets and corresponding images in the Gödel set.

E.g., $a_2^- = \mathbf{0}$ and $a_2^+ = \mathbf{1}$ as $\mathbf{0} = a_0 \prec a_2 \prec a_1 = \mathbf{1}$, hence $h(a_2) = \frac{1}{2}$. It is easy to verify that h has the required properties. □

For the construction of Gödel sets from Kripke frames, the Horn monomorphism h from the previous Lemma has, in addition to preserving existing infima and suprema, to preserve 'cuts': Let $\mathcal{P} = (P, \preceq)$ be a linear order. For $X \subseteq P$ we define $X^\uparrow := \{y \in P \colon \exists x \in X x \preceq y\}$ and $X^\downarrow := \{y \in P \colon \exists x \in X y \preceq x\}$. A cut of $\mathcal{P}$ is a pair of subsets (X, Y) of P such that $X^\downarrow \cap Y^\uparrow = \emptyset$, $X \cup Y = P$, and neither the (order theoretic) supremum of X, denoted $\sup X$, nor the (order theoretic) infimum of Y, denoted $\inf Y$, exist. Preserving cuts then means that for any cut (X, Y) of $\mathcal{P}$ the supremum of $h(X)$ in $\mathbb{R}$, denoted $\sup h(X)$, and the infimum of $h(Y)$ in $\mathbb{R}$, denoted $\inf h(Y)$, coincide (here, $h(X)$ denotes the set $\{h(x) \colon x \in X\}$). The following Lemma states that the Horn monomorphism h has this property of preserving cuts in a slightly more general way which is needed in the proof of Theorem 18.

Lemma 17. *Let $\mathcal{P} = (P, \preceq)$ be a countable linear order and let X and Y be subsets of P such that $X^\downarrow \cap Y^\uparrow = \emptyset$, $X^\downarrow \cup Y^\uparrow = P$, and suppose that neither $\sup X$ nor $\inf Y$ exists. Let h be the Horn monomorphism from the proof of Lemma 16. Then $\sup h(X) = \inf h(Y)$.*

Proof. First observe that, w.l.o.g., we can assume $X = X^\downarrow$ and $Y = Y^\uparrow$, because $\sup h(X) = \sup h(X^\downarrow)$, $\inf h(Y) = \inf h(Y^\uparrow)$, and the existence of $\sup X^\downarrow$ implies the existence of $\sup X$, similar for $\inf Y^\uparrow$ versus $\inf Y$.

If one of X or Y is empty, then the assertion follows as $\sup h(P) = 1$ and $\inf h(P) = 0$ by construction of h. Assume now that $X \neq \emptyset$ and $Y \neq \emptyset$. Consider the construction given in the proof of Lemma 16. At step n, let $x_n := \max(X \cap \{a_i \colon i \leq n\})$ and $y_n := \min(Y \cap \{a_i \colon i \leq n\})$. Then $h(y_n) - h(x_n)$ converges to 0 because $(x_n)_n$ cannot become constant as $\sup X$ does not exist, similar for $(y_n)_n$. Furthermore, each time a value changes, say $x_n \prec x_{n+1}$, the distance gets cut by at least 1/2: $h(y_{n+1}) - h(x_{n+1}) \leq \frac{h(y_n) - h(x_n)}{2}$. Also, the assumption $X^\downarrow \cap Y^\uparrow = \emptyset$ implies that $\sup h(X) \leq \inf h(Y)$. Thus, $\sup h(X) = \inf h(Y)$. □

Note that the continuity condition in the last Lemma in particular implies that h preserves all existing suprema and infima in K and that $\inf h(W) = 0$ and $\sup h(W) = 1$.

Theorem 18. *For every countable linear Kripke frame K there is a Gödel set V_K such that $\mathbf{L}(K) = \mathbf{G}_{V_K}$.*

Proof. Let $K = (W, \preceq)$ be a countable linear Kripke frame. The construction of V_K will be in three steps: First, we will enlarge K by doubling all limit worlds; then we will apply the Horn monomorphism (Lemma 16 and 17) to embed the enlarged Kripke frame to $\mathbb{Q} \cap [0, 1]$; finally, V_K will be the completion of the range of this embedding.

Let W^* be a disjoint copy of W whose elements can be accessed by the bijection $^*\colon W \to W^*$. Elements of W^* serve as names for points which we may have to add. We extend $\preceq$ to a total order $\preceq^*$ on $W \cup W^*$ by putting w^* as the direct successor of w for each $w \in W$, see Fig. 2.

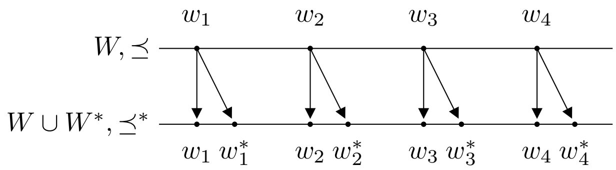

Figure 2: Extending $(W, \preceq)$ to $(W \cup W^*, \preceq^*)$.

Formally we define $\preceq^*$ as follows:

$$\preceq^* \quad := \quad \preceq \cup \{(v^*, w^*)\colon v \preceq w\} \cup \{(v, w^*)\colon v \preceq w\} \cup \{(v^*, w)\colon v \prec w\} \quad .$$

Let $\mathrm{Lim}(W)$ denote the set of limit worlds in W:

$$w \in \mathrm{Lim}(W) \quad \text{iff} \quad (\forall w' \succ w)(\exists w'' \succ w)(w'' \prec w') \ .$$

Observe that a maximal element of K, if it exists, would be in $\mathrm{Lim}(W)$. We define

$$W' \quad := \quad W \cup \{w^*\colon w \in \mathrm{Lim}(W)\}$$

and we define $\preceq'$ as the restriction of $\preceq^*$ to W':

$$\preceq' \quad := \quad \preceq^* \cap (W' \times W') \ .$$

Let $K' := (W', \preceq')$.

Next, we apply the Horn monomorphism from the proof of Lemma 16 to the converse of K', i.e. to $(W', \succeq')$. We obtain an embedding σ from $(W', \succeq')$ to $(\mathbb{Q} \cap [0, 1], \leq)$ which, by Lemma 17, satisfies the following form of continuity: for any subsets X and Y of W', if $\{w \in W'\colon \exists x \in X w \preceq' x\} \cap \{w \in W'\colon \exists y \in Y y \preceq' w\} = \emptyset$ and $\{w \in W'\colon \exists x \in X w \preceq' x\} \cup \{w \in W'\colon \exists y \in Y y \preceq' w\} = W'$ then $\sup \sigma(Y) = \inf \sigma(X)$.

To finish our construction, let V_K be the closure of $\sigma(W')$:

$$V_K \quad := \quad \overline{\sigma(W')} \ .$$

We now have to show that $\mathbf{L}(K) = \mathbf{G}_{V_K}$, which we prove by constructing an isomorphism f between $(\mathrm{Up}(K), \subseteq)$ and $(V_K, \leq)$. Lemma 10 together with Proposition 14 then show that the logics are the same:

$$\mathbf{L}(K) = \mathbf{L}(\mathrm{Up}(K)) = \mathbf{G}_{V_K}.$$

Let $s \in \mathrm{Up}(K)$. We define f by case distinction on the type of s:

1. $\mathrm{tp}(s) = \alpha$, then s contains a minimal element w: Let $f(s) = \sigma(w)$.

2. $\mathrm{tp}(s) = \beta$, then s^c contains a maximal element w: Let $f(s) = \sigma(w^*)$.

3. $\mathrm{tp}(s) = \gamma$: Let $f(s) = \sup \sigma(s)$.

Claim. f is an isomorphism between $(\mathrm{Up}(K), \subseteq)$ and $(V_K, \leq)$.

We now prove the claim. First observe that f is obviously well-defined. Also observe in case γ that s and s^c satisfy the preconditions for the continuity of σ: Obviously $\{w \in W' : (\exists x \in s^c) w \preceq' x\} \cap \{w \in W' : (\exists y \in s) y \preceq' w\} = \emptyset$, but also $\{w \in W' : (\exists x \in s^c) w \preceq' x\} \cup \{w \in W' : (\exists y \in s) y \preceq' w\} = W'$ and neither s has an infimum nor s^c has a supremum in W' by construction of W'. Hence:

$$\mathrm{tp}(s) = \gamma \quad \Rightarrow \quad f(s) = \sup \sigma(s) = \inf \sigma(s^c) \tag{1}$$

To show that f is injective assume that $f(s_1) = f(s_2)$. We consider all combinations of types of s_1 and s_2.

The cases where $\{\mathrm{tp}(s_1), \mathrm{tp}(s_2)\} \subseteq \{\alpha, \beta\}$ are trivial as σ is injective.

Consider now the case that s_1 is of type α or β, and s_2 is of type γ. Then $f(s_1) = \sigma(w)$ for some $w \in W'$, and $f(s_2) = \inf \sigma({s_2}^c) = \sup \sigma(s_2)$ by (1). But then $(\forall w' \in s_2)(w \preceq' w')$ and $(\forall w' \in {s_2}^c)(w' \preceq' w)$, which contradicts the condition that s_2 is of type γ: If $w \in W$ then $w \in s$ or $w \in s^c$; the former gives a minimal element in s_2, the latter a maximal one in s_2^c, which clearly contradicts the assumption that s_2 is of type γ. In the case that $w = \bar{w}^*$ for some $\bar{w} \in W$ we obtain that $\bar{w}$ must be the maximal element in s^c, which again contradicts that s_2 is of type γ.

Considering the case that both s_1 and s_2 are of type γ, then $f(s_1) = \sup \sigma(s_1) = \inf \sigma(s_1)$ and $f(s_2) = \sup \sigma(s_2) = \inf \sigma(s_2)$ by (1). By construction we obtain that $s_1 \cap s_2^c = \emptyset$, $s_2 \cap s_1^c = \emptyset$ and hence $s_1 = s_2$.

In order to show that f is surjective, let $v \in V_K$. We have to find a pre-image s of v under f. If $v \in \sigma(W')$ then this is easy: If $v = \sigma(w)$ for some $w \in W$ then let $s := w^\uparrow$; if $v = \sigma(w^*)$ for some $w \in W$ then let $s := w^\uparrow \setminus \{w\}$.

Now let us assume that $v \in \overline{\sigma(W')} \setminus \sigma(W')$. Then there exists a strictly increasing or a strictly decreasing sequence in $\sigma(W')$ with limit v.

If there is a strictly increasing sequence let $s = \{w \in W : \sigma(w) < v\}$. It is easy to verify that s is of type γ, hence $f(s) = v$. Otherwise, there exists only a strictly decreasing sequence in $\sigma(W')$ with limit v. Note that if there is only a strictly decreasing sequence in $\sigma(W')$ but no strictly increasing sequence, the limit of this sequence can only be 0 as in all other cases the limit would be an element of $\sigma(W')$. Thus $v = \inf \sigma(W') = 0$ and we have $f(\emptyset) = 0$.

Thus we have shown that f is a bijection between $\mathrm{Up}(K)$ and V_K. Observe that f is a $\subseteq$–$\leq$-homomorphism as σ is an embedding. Hence f is a $(\mathrm{Up}(K), \subseteq)$–$(V_K, \leq)$ isomorphism as required. □

The following example considers the logic of the Kripke frame with set of worlds $\mathbb{Q}$. Takano in [Tak87b] has shown that this logic is axiomatised by any complete axiom system for first-order intuitionistic logic (see e.g. [Tro77]) plus the axiom scheme of linearity $(A \to B) \vee (B \to A)$ and the axiom scheme of constant domain (or quantifier shift) $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$, where x must not occur free in A. This axiomatisation is the same as the one for the standard first-order Gödel logic, i.e. the one based on the full interval $[0, 1]$ (cf. [Hor69]). Hence, we can expect that our construction derives a related Gödel set from the Kripke frame $\mathbb{Q}$.

Example 19 (The logic $\mathbf{L}(\mathbb{Q})$). Let $K_{\mathbb{Q}} = (\mathbb{Q}, \leq)$ be the Kripke frame of $\mathbb{Q}$. We want to describe the Gödel set $V_{\mathbb{Q}}$ corresponding to $K_{\mathbb{Q}}$ which is obtained by the construction given in the proof of the previous Theorem. $V_{\mathbb{Q}}$ will be isomorphic to the set of upsets of $\mathbb{Q}$.

Note that for every element $q \in \mathbb{Q}$ there are two designated upsets in $\mathrm{Up}(K_{\mathbb{Q}})$, $q^\uparrow$ and $q^\uparrow \setminus \{q\}$. Between these two upsets there is no other upset in $\mathrm{Up}(K_{\mathbb{Q}})$.

Thus, $q^\uparrow$ and $q^\uparrow \setminus \{q\}$ under the isomorphism between $\mathrm{Up}(K_{\mathbb{Q}})$ and $V_{\mathbb{Q}}$ determine an open interval of $[0,1]$ which will never contain a point during our construction. Hence, doing this for all elements of $\mathbb{Q}$, countably many disjoint open intervals are generated which are densely ordered, which is achieved by a set isomorphic to the Cantor set.

To be more precise: For every $q \in \mathbb{Q}$ the upset $q^\uparrow \setminus \{q\}$ is of type β. Thus, our construction from the last proof duplicates all the rational number, i.e. $\mathbb{Q}' = \mathbb{Q} \cup \{q^* : q \in \mathbb{Q}\}$ and $\leq' = \leq^*$. Now fix a particular enumeration of $\mathbb{Q} = \{q_1, q_2, \dots\}$ and consider the following enumeration induced on $\mathbb{Q}' = \{q_1, q_1^*, q_2, q_2^*, \dots\}$. The images of the pairs q_1, q_1^*, q_2, q_2^*, etc., under the Horn function h determine a sequence of disjoint open intervals of $[0,1]$ which are removed from $[0,1]$. This obviously mimics Cantors middle third construction of repeatedly removing the middle thirds of line segments of $[0,1]$. Hence the image of $\mathbb{Q}'$ under the Horn function h for this enumeration is a set isomorphic to the set of boundary points of the Cantor set, and the completion of $h(\mathbb{Q}')$ is a set isomorphic to the Cantor set. This situation is displayed in Fig. 3 where the x-ed intervals indicate that no point of $\mathbb{Q}'$ is mapped into such an interval by h.

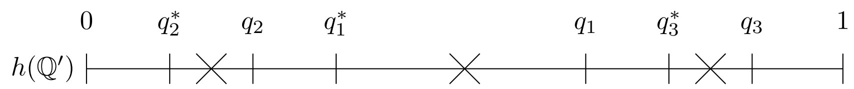

Figure 3: The image of $\mathbb{Q}'$ under the Horn function h

Now, the Gödel logic $\mathbf{G}_{\mathbb{C}_{[0,1]}}$ generated by the Cantor set $\mathbb{C}_{[0,1]}$ is equal to the Gödel logic of the full interval, $\mathbf{G}_{[0,1]}$ (cf. [Pre03, BPZ]). To obtain an idea for this, first observe that obviously $\mathbf{G}_{[0,1]} \subseteq \mathbf{G}_{\mathbb{C}_{[0,1]}}$. Furthermore, for each $\varphi \notin \mathbf{G}_{[0,1]}$ we can find a valuation based on a countable model which makes φ false; hence the occurring truth values form a countable set (not necessarily closed!) which can be embedded into $\mathbb{C}_{[0,1]}$ such that existing infima and suprema are preserved. This gives rise to an interpretation based on $\mathbb{C}_{[0,1]}$ which also makes φ false. Hence, also $\varphi \notin \mathbf{G}_{\mathbb{C}_{[0,1]}}$.

3.2 From Gödel sets to Kripke frames

We start by recalling some definitions from topology and descriptive set theory for Polish spaces (i.e. complete, separable metric spaces): A point x is called a *limit point* if in every open neighbourhood U of x there is a point $y \in U$ with $x \neq y$. A Polish space is called *perfect* if all its points are limit points. A subset P of a Polish space is called *perfect* if it is closed and all points in P are limit points. Finally recall the following Theorem of Cantor-Bendixon on the representation of Polish spaces (for a detailed exposition see [Kec95, I.6] or [Mos80, 2A.1]):

Theorem 20 (Cantor-Bendixon). *Let X be a closed subset of a Polish space. Then X can be uniquely written as $X = P \cup C$, with $P \cap C = \emptyset$, P a perfect subset of X, and C countable. P is called the* perfect kernel *of X (denoted by X^∞), and C is called the scattered part of X.*

We will only use the following specialisation of the previous Theorem:

Corollary 21. *If V is a closed subset of $\mathbb{R}$ then it can be uniquely written as $V = P \cup C$ where P is perfect, C is countable and $P \cap C = \emptyset$.*

Going from Gödel sets to Kripke frames is not as complicated as the other direction. First we consider countable Gödel sets. For the general case of uncountable Gödel sets we will use Example 19 and a splitting lemma (Lemma 24) which divides uncountable Gödel sets into a countable part and a part containing a perfect set.

Lemma 22. *For every countable Gödel set V there is a countable linear Kripke frame K_V such that $\mathbf{G}_V = \mathbf{L}(K_V)$.*

Proof. Since V is countable and closed, it can be viewed as a complete and completely distributive linear lattice. Every element of V is either an isolated point, or it is the limit of some isolated points. Thus every element of V is the join of a set of completely join-irreducible elements and V is isomorphic to a complete linear ring of sets (see [Ran52] for definitions of join-irreducibility and this result). Furthermore, a lattice is isomorphic to a complete ring of sets if and only if it is isomorphic to the lattice of order ideals of some partial order P (see e.g. [Dav79] for the definition of order ideals and this result). Utilising Lemma 10, the logic $\mathbf{G}_V$ and $\mathbf{L}(P)$ coincide. □

Remark. It is worth explicitly describing the construction of the Kripke frame underlying the previous proof. This is useful for finding Kripke frames for concretely given Gödel sets such that the logics defined by the Kripke frames are the same as the logics defined by the Gödel sets.

Let V be a countable Gödel set. By removing proper suprema from V we obtain a corresponding Kripke frame K_V: Let $\mathrm{Sups}(V)$ be the set of all suprema of V,

$$\mathrm{Sups}(V) \;:=\; \{p \in V : \exists (p_n) \subset V \text{ strictly increasing to } p\} \cup \{0\}.$$

We define the set of worlds as $W_V := V \setminus \mathrm{Sups}(V)$. Then the Kripke frame $K_V := (W_V, \geq)$ defines the same logic as the Gödel set V. This construction works because a supremum of V will reoccur in $\mathrm{Up}(K_V)$ as the upset of all elements smaller than that supremum.

A variant of the next Proposition was first proven in [Pre03] to characterise axiomatisability of Gödel logics. The current form is taken from [BGP].

Proposition 23. *Let V be a Gödel set with non-empty perfect kernel P and let $W = V \cup [\inf P, 1]$. Then the logics induced by V and W are the same, i.e. $\mathbf{G}_V = \mathbf{G}_W$.*

For the treatment of general, i.e. uncountable, Gödel sets we need the following splitting Lemma which allows to split Kripke frames into parts and consider the logics of these parts only.

Lemma 24. *Let V_1 and V_2 be Gödel sets and $K_1 = (W_1, \preceq_1)$ and $K_2 = (W_2, \preceq_2)$ be Kripke frames such that $(V_i, \leq)$ and $(\mathrm{Up}(K_i), \subseteq)$ are isomorphic. Assume $W_1 \cap W_2 = \emptyset$. Let $\alpha \in (0, 1)$, define*

$$V \;:=\; \alpha V_1 \cup ((1-\alpha)V_2 + \alpha)$$

and $K := (W_2 \cup W_1, \preceq)$ with

$$\preceq \;:=\; \preceq_2 \cup \preceq_1 \cup \{(w_2, w_1) : w_2 \in W_2, w_1 \in W_1\} \quad ,$$

see Fig. 4. Then $(V, \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic, too.

Proof. Let f_i be the isomorphism from V_i to $\mathrm{Up}(K_i)$. We define $f : V \to \mathrm{Up}(K)$ as follows: If $v \in [0, \alpha] \cap V$ then $f(v) = f_1(v/\alpha)$. If $v \in [\alpha, 1] \cap V$ then $f(v) = W_1 \cup f_2((v - \alpha)/(1 - \alpha))$.

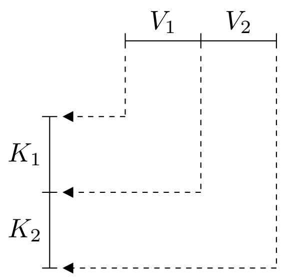

Figure 4: The relation of V_1, V_2 to K_1, K_2.

First observe that f is well defined: the only critical point is at α where we have two ways to compute $f(\alpha)$:

$$f(\alpha) = f_1(\alpha/\alpha) = f_1(1) = W_1$$

and

$$f(\alpha) = W_1 \cup f_2((\alpha - \alpha)/(1 - \alpha)) = W_1 \cup f_2(0) = W_1 \cup \emptyset = W_1 \ .$$

It is easy to verify that f is a $(V, \leq)$–$(\mathrm{Up}(K), \subseteq)$ isomorphism: f being bijective is reduced to f_1 and f_2 being bijective, and it is also immediate from the construction that f is a $\leq$–$\subseteq$ homomorphism. □

Theorem 25. *For every Gödel set V there is a countable linear Kripke frame K_V such that $\mathbf{G}_V = \mathbf{L}(K_V)$.*

Proof. In Corollary 21 the Cantor-Bendixon representation of V gives a countable set C and a perfect set P such that $V = C \cup P$ and $C \cap P = \emptyset$. If V is countable, then P is empty and the Gödel logic induced by V can already be represented using Lemma 22. So assume that V is not countable, which means P is not empty. Let $\alpha := \inf P$, $V'' := V \cup [\alpha, 1]$ and $V' := (V \cap [0, \alpha]) \cup \mathbb{C}_{[\alpha,1]}$, where $\mathbb{C}_I$ is the Cantor middle-third set on the interval I, which is a perfect set. Using Proposition 23 we obtain that $\mathbf{G}_V = \mathbf{G}_{V''} = \mathbf{G}_{V'}$. Hence, it is enough to consider V'.

In the case that $\alpha = 0$ we have $V' = \mathbb{C}_{[0,1]}$, in which case the Gödel logic based on V' is the same as the $\mathbf{L}(\mathbb{Q})$, see Example 19.

Otherwise let $V_1 := (1/\alpha)(V \cap [0, \alpha])$ and $V_2 := \mathbb{C}_{[0,1]}$. Then we can write V' as

$$V' = \alpha V_1 \cup ((1 - \alpha)V_2 + \alpha).$$

By construction of α, V_1 is countable and due to V being closed V_1 is also closed. Hence, by the proof of Lemma 22 we can find a countable linear Kripke frame K_1 such that $(V_1, \leq)$ and $(\mathrm{Up}(K_1), \subseteq)$ are isomorphic. Due to Example 19 we know that $(V_2, \leq)$ and $(\mathrm{Up}(K_{\mathbb{Q}}), \subseteq)$ are isomorphic. Applying Lemma 24 we obtain a countable Kripke frame K such that $(V', \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic. Finally, using Proposition 14 and Lemma 10, we obtain for the induced logics

$$\mathbf{G}_V = \mathbf{G}_{V'} = \mathbf{L}(\mathrm{Up}(K)) = \mathbf{L}(K) \ .$$

□

It is worth pointing out some structural consequences which can be inferred from our constructions. Let K be a countable linear Kripke frame and let V_K be the corresponding Gödel set. K having a top element is equivalent to 0 being isolated in V_K, and K having a bottom element is equivalent to 1 being isolated in V_K. Let K *ends with* $\mathbb{Q}$ denote that there is an embedding σ of $\mathbb{Q}$ into K such that $\forall k \in K\ \exists q \in \mathbb{Q}\ k \preceq \sigma(q)$. In this case we have that $\mathbf{L}(K) = \mathbf{L}(\mathbb{Q})$. To see this observe that, as in Example 19, the condition 'K ends with $\mathbb{Q}$' implies that V_K contains a Cantor set which contains 0. But then Proposition 23 shows that the induced Gödel logic $\mathbf{G}_{V_K}$ is the same as the Gödel logic of the full unit interval, hence

$$\mathbf{L}(K) = \mathbf{G}_{V_K} = \mathbf{G}_{[0,1]} = \mathbf{L}(\mathbb{Q}) \ .$$

It is interesting to note that Theorem 25 cannot be deduced from the Löwenheim-Skolem Theorems in [Ono73] and Lemma 22. Rather, the results presented in the present paper indicate that the Löwenheim-Skolem Theorem in [Ono73, Theorem 4.8], which deals with reducing the cardinality of the pseudo-Boolean algebra, cannot be strengthened in the form that it is reduced to the cardinality of the universe (assuming it is infinite), i.e. in terms of [Ono73, Theorem 4.8], $\lambda' = 2^\lambda$ cannot be replaced by λ in general. To see this observe that the pseudo-Boolean algebra $[0, 1]$ cannot be replaced by any countable pseudo-Boolean algebra: the Gödel logic of the former is axiomatisable (see above), where the Gödel logic of any countable truth value set is not axiomatisable (see Proposition 26).

4 Applications and conclusions

Due to the strong correspondence results given in Theorems 18 and 25 we can transfer results recently obtained on Gödel logics to logics of countable linear Kripke frames with constant domains.

First let us consider axiomatisability. Only recently a complete characterisation of axiomatisability of first order Gödel logics has been given in [Pre03] and the follow up [BPZ], which can be used to give a complete characterisation of axiomatisability of countable linear Kripke frames with constant domains.

The following proposition characterises the axiomatisability of Gödel logics in terms of the topological structure of the respective truth value set:

Proposition 26 ([Pre03, BPZ]). *Let V be a Gödel set and P the perfect kernel of V. $\mathbf{G}_V$ is axiomatisable iff either (i) V is finite, or (ii) $0 \in P$, or (iii) $P \neq \emptyset$, and 0 isolated in V (thus $0 \notin P$).*

In all the other cases (V countable; 0 not isolated and not contained in the perfect kernel) the respective logics are not recursively enumerable.

Combining Proposition 26 with the results of this paper we obtain the following two Corollaries characterising the axiomatisability of countable linear Kripke frames with constant domains.

Corollary 27. *Let K be a countable linear Kripke frame. The intermediate predicate logic defined by K on constant domains is axiomatisable if and only if K is finite, or if $\mathbb{Q}$ can be embedded into K, and either K has a top element or ends with a copy of $\mathbb{Q}$.*

Proof. Let K be a countable linear Kripke frame, V_K the corresponding Gödel set and P the perfect kernel of V_K. By Proposition 26 we know that $\mathbf{G}_{V_K} = \mathbf{L}(K)$ is finitely axiomatisable iff (i) V_K is finite, or (ii) $0 \in P$, or (iii) $P \neq \emptyset$, $0 \notin P$, and 0 isolated in V_K. For K these conditions are equivalent to: (i) K is finite, or (ii) K ends with $\mathbb{Q}$, or (iii) $\mathbb{Q}$ can be embedded into K, and K has a top element. □

Corollary 28. *Let K be a countable linear Kripke frame. If K is either not finite and $\mathbb{Q}$ cannot be embedded into K (i.e., K is scattered), or $\mathbb{Q}$ can be embedded into K, but K does not end with $\mathbb{Q}$ and K does not have a top element, then the intermediate predicate logic defined by K on constant domains is not recursively enumerable.*

Finally, we consider the number of different logics. A reasonable argumentation for a lower bound on it would be as follows: If we have a basic logic with extensions in which each of countable many principles can be either true or false, then we would expect uncountably many different logics. As an example let us consider the class of all intermediate predicate logics, i.e. all those logics which are between Intuitionistic Logic and Classical Logic (cf. [Ono73]). Here, we have a common basic logic, Intuitionistic Logic, and extensions of it by different principles. And in fact there are uncountably many intermediate predicate logics. Another example is the class of modal logics which has K as its common basic logic.

Considering Gödel logics, there is a common basic logic, the logic of the full interval, which is included in all other Gödel logics. On the side of logics defined by linear Kripke frames on constant domains this corresponds to the logic determined by a set of worlds of order-type $\mathbb{Q}$. There are still countably many extension principles but, surprisingly, in total only countably many different logics. This has been proven recently by formulating and solving a variant of a Fraïssé Conjecture [Fra48] on the structure of countable linear orderings w.r.t. continuous embeddability:

Proposition 29 ([BGP]). *The set of Gödel logics is countable.*

Again, this result can be transfered to the logics defined by countable linear Kripke frames on constant domains using the results from this paper.

Corollary 30. *The set of intermediate predicate logics defined by countable linear Kripke frames on constant domains is countable.*

Another surprising aspect from the point of view of the last Corollary is that while there are uncountably many different countable linear orderings (which can be taken as Kripke frames), the class of logics defined by them on constant domains only contains countably many elements. Furthermore, the last result is contrasted by the fact that the number of all intermediate logics extending the basic linear logic with constant domains is uncountable.

There are many more results which could be mentioned here, but we consider the presented results as the most interesting and surprising ones.

Acknowledgements

We thank the anonymous referee for many valuable remarks, and for advise which significantly improved the presentation of this paper. We also thank Ken Johnson and Andrew Lewis for carefully proof reading a previous version of this paper.

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