Applications of cut–free infinitary derivations to generalized recursion theory

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1 Introduction

Cut elimination is the main tool in proof theoretical research. The emphasis there, however, is on the cut–elimination procedure, i.e., the dynamical aspect of cut–elimination. In this paper we want to show that also the statical aspect, i.e., the mere existence of cut–free derivations, has consequences which may be viewed to belong to Descriptive Set Theory or Generalized Recursion Theory.

We will introduce an infinitary proof system for a language of second order arithmetic which is complete for Π^1_1–sentences. This leads to a notion of truth complexity for Π^1_1–sentences. We define the truth complexity as the depth of the shortest infinitary derivation for φ. The main observation there is the Boundedness Theorem telling that the order–type of a Σ^1_1–definable well–ordering $\prec$ is less than or equal to the truth complexity of $TI(\prec)$ where $TI(\prec)$ expresses transfinite induction along $\prec$. The fact that for every Σ^1_1–collection of well–orderings of order types $< \omega_1^{CK}$ there is an ordinal $< \omega_1^{CK}$ which bounds all these order types is commonly known as the Boundedness Principle of Generalized Recursion Theory. We will obtain it as a consequence of our proof theoretical Boundedness Theorem which in turn is a consequence of cut–freeness. This yields a proof of the Boundedness Principle which does not use the Analytical Hierarchy Theorem.

We are indebted to the referee for many valuable remarks.

2 The infinitary proof system

Let $\mathcal{L}_2$ denote the language of Second Order Number Theory including constants for all primitive–recursive functions and relations. We introduce the language $\mathcal{L}_2^\infty$ of infinitary propositional logic with second order quantifiers. The non–logical symbols of $\mathcal{L}_2^\infty$ are those of $\mathcal{L}_2$. The *logical symbols* are $=$, $\neq$, $\in$, $\notin$, $\bigwedge$, $\bigvee$, $\forall$, $\exists$, and second order variables X,Y,Z,X_1, $\dots$.

- *Terms are built from* $\underline{0}$ *and constants for primitive–recursive functions. The symbol* $\underline{n}$ *abbreviates* $\underbrace{(S \dots (S}_{n\text{–}times}\,\underline{0}))$ *where* S *is a symbol for the successor function. Observe that for any term* t *its value* $t^{\mathbb{N}}$ *can be primitive–recursively computed.*
- *Atomic formulas are* $(s = t), (s \neq t), (s \in X), (s \notin X), (Rt_1 \dots t_n)$ *where* $s, t, t_1, \dots, t_n$ *are terms and* R *is a symbol for an* n*–ary primitive–recursive relation.*
- *If* $\langle \phi_i | i \in I \rangle$ *for* $\emptyset \neq I \subseteq \mathbb{N}$ *is a sequence of* $\mathcal{L}_2^\infty$*–formulas then* $\bigwedge\limits_{i \in I} \phi_i$ *and* $\bigvee\limits_{i \in I} \phi_i$ *are* $\mathcal{L}_2^\infty$*–formulas.*
- *If* $\varphi(X)$ *is an* $\mathcal{L}_2^\infty$*–formula then* $\forall X \varphi(X)$ *and* $\exists X \varphi(X)$ *are* $\mathcal{L}_2^\infty$*–formulas.*

We call an $\mathcal{L}_2^\infty$ formula *first order* iff it does not contain quantifiers. We denote the set of first order formulas by $\mathcal{L}_1^\infty$.

The semantics for $\mathcal{L}_2^\infty$ is given in the natural way, where $\exists X$ and $\forall X$ are supposed to range over *all* subsets of $\mathbb{N}$. We write $\mathbb{N} \models \varphi$ to denote that the sentence φ is standardly valid.

We define the negation $\neg\varphi$ of an $\mathcal{L}_2^\infty$–formula inductively by

- $\neg(s = t) :\equiv (s \neq t)$, $\neg(s \neq t) :\equiv (s = t)$,
- $\neg(t \in X) :\equiv (t \notin X)$, $\neg(t \notin X) :\equiv (t \in X)$,
- $\neg(Rt_1, \dots, t_n) :\equiv (\overline{R}t_1, \dots, t_n)$ *where* $\overline{R}$ *is the symbol denoting the complement of the relation denoted by* R.
- $\neg(\bigwedge\limits_{i \in I} \phi_i) :\equiv \bigvee\limits_{i \in I} \neg\phi_i$, $\neg(\bigvee\limits_{i \in I} \phi_i) :\equiv \bigwedge\limits_{i \in I} \neg\phi_i$.
- $\neg(\forall X \varphi(X)) :\equiv (\exists X \neg\varphi(X))$, $\neg(\exists X \varphi(x)) :\equiv (\forall X \neg\varphi(X))$.

There is a canonical translation * of the formulas of $\mathcal{L}_2$ not containing free number variables into $\mathcal{L}_2^\infty$ by putting

- $\varphi^* :\equiv \varphi$ *for atomic* φ
- $(\varphi \wedge \psi)^* :\equiv \bigwedge\{\varphi^*, \psi^*\}$, $(\varphi \vee \psi)^* :\equiv \bigvee\{\varphi^*, \psi^*\}$
- $(\neg\varphi)^* :\equiv \neg\varphi^*$
- $(\forall x\varphi(x))^* :\equiv \bigwedge_{n\in\omega}\varphi(\underline{n})^*$, $(\exists x\varphi(x))^* :\equiv \bigvee_{n\in\omega}\varphi(\underline{n})^*$
- $(\forall X\varphi(X))^* :\equiv \forall X\varphi(X)^*$, $(\exists X\varphi(X))^* :\equiv \exists X\varphi(X)^*$.

We obviously have

$$\mathbb{N} \models \varphi \;\Leftrightarrow\; \mathbb{N} \models \varphi^*$$

for all $\mathcal{L}_2$–sentences φ. In the rest of the paper we will mostly identify φ and φ^*. It will usually be clear from the context if φ is the $\mathcal{L}_2$ or $\mathcal{L}_2^\infty$ formula.
The diagram of $\mathbb{N}$ contains all true atomic sentences of $\mathcal{L}_2^\infty$. Observe that Diagram($\mathbb{N}$) is a recursive set.

2.1 Definition Let Δ be a finite set of $\mathcal{L}_2^\infty$–formulas and α an ordinal. We define the infinitary proof relation $\models^{\alpha} \Delta$ inductively by the following clauses.

(Ax $\mathbb{N}$) *If* $\Delta \cap$ *Diagram* $(\mathbb{N}) \neq \emptyset$ *then* $\models^{\alpha} \Delta$ *for any* α .

(Ax L) *If* $s^{\mathbb{N}} = t^{\mathbb{N}}$ *then* $\models^{\alpha} \Delta, s \notin X, t \in X$ *for any* α.

($\bigwedge$) *If* $\models^{\alpha_i} \Delta, \varphi_i$ *and* $\alpha_i < \alpha$ *holds for all* $i \in I$ *then* $\models^{\alpha} \Delta, \bigwedge\limits_{i\in I}\varphi_i$.

($\bigvee$) *If* $\models^{\alpha_0} \Delta, \varphi_i$ *and* $\alpha_0 < \alpha$ *holds for some* $i \in I$ *then* $\models^{\alpha} \Delta, \bigvee\limits_{i\in I}\varphi_i$.

($\forall$) *If* $\models^{\alpha_0} \Delta, \varphi(X)$ *and* $\alpha_0 < \alpha$ *holds for some set variable* X *not occurring in* $\Delta, \forall Z\varphi(Z)$ *then* $\models^{\alpha} \Delta, \forall Z\varphi(Z)$

($\exists$) *If* $\models^{\alpha_0} \Delta, \varphi(X)$ *and* $\alpha_0 < \alpha$ *then* $\models^{\alpha} \Delta, \exists Z\varphi(Z)$.

There are some basic properties which follow easily by induction on α.

$$\models^{\alpha} \Delta, \alpha \leq \beta, \Delta \subseteq \Gamma \;\Rightarrow\; \models^{\beta} \Gamma. \tag{1}$$

$$\models^{\alpha} \Delta, \bigwedge_{i\in I}\phi_i \;\Rightarrow\; \models^{\alpha} \Delta, \phi_i \text{ for all } i \in I. \tag{2}$$

$$\models^{\alpha} \Delta, \bigvee_{i=1,\ldots,n}\varphi_i \;\Rightarrow\; \models^{\alpha} \Delta, \varphi_1, \ldots, \varphi_n. \tag{3}$$

$$\models^{\alpha} \Delta(s) \text{ and } s^{\mathbb{N}} = t^{\mathbb{N}} \;\Rightarrow\; \models^{\alpha} \Delta(t). \tag{4}$$

$$\overset{\alpha}{\models} \Delta, \forall X \varphi(X) \;\Rightarrow\; \overset{\alpha}{\models} \Delta, \varphi(Z) \text{ for any } Z. \tag{5}$$

We refer to (1) as *structural rule*, to (2) as $\bigwedge$*–inversion*, to (3) as $\bigvee$*–exportation*, to (4) as *equality–rule* and to (5) as $\forall$*–inversion*. If an infinitary derivation does neither contain $\forall$– nor $\exists$–rules we talk about a first order derivation. The existence of a first order derivation is denoted by ${}^1\overset{\alpha}{\models} \Delta$.

2.2 Lemma *If Δ is a set of first order formulas and $\overset{\alpha}{\models} \Delta$, then ${}^1\overset{\alpha}{\models} \Delta$.*

The proof is immediate by induction on α. The infinitary calculus is obviously sound. By induction on α we get the following lemma.

2.3 Lemma $\overset{\alpha}{\models} \varphi_1, \ldots, \varphi_n \;\Rightarrow\; \mathbb{N} \models \varphi_1 \vee \ldots \vee \varphi_n$.

The opposite direction of Lemma 2.3 is in general not true. But we can save it in special situations.

2.4 Definition Call a formula φ in $\mathcal{L}_2^\infty$ a Π*–formula* if

$$\varphi \equiv \forall X_1 \ldots \forall X_n \psi(X_1, \ldots, X_n)$$

and $\psi(X_1, \ldots, X_n)$ is first order.

2.5 Theorem *For any Π–sentence φ we have*

$$\mathbb{N} \models \varphi \;\Leftrightarrow\; (\exists \alpha < \omega_1) \overset{\alpha}{\models} \varphi$$

where ω_1 denotes the first uncountable ordinal.

Proof: The direction from right to left is Lemma 2.3. A detailed proof of the opposite direction is in [5] Theorem 9.6 which holds for first order derivations. However, to make this paper self contained, we repeat a sketch of the proof.
A tree is a set of sequence numbers which is closed under initial segments. For sequence numbers s_0 and s_1 we denote by $s_0 \subseteq s_1$ that s_0 codes an initial segment of s_1.

2.6 Definition We are going to define search trees for finite sequences of $\mathcal{L}_1^\infty$–formulas. Such a sequence is called *reducible* if it contains at least one non atomic formula. The left most non atomic formula in a reducible sequence is called *distinguished*. The *reduced sequence* Δ^r of a reducible sequence Δ is obtained by removing the distinguished formula from the sequence.
The search tree for a finite sequence Δ of $\mathcal{L}_1^\infty$-formulas is a tree S_Δ together with a label function which assigns a finite sequence $\delta(s)$ of $\mathcal{L}_1^\infty$-formulas to each node $s \in S_\Delta$. It is defined by the following clauses:

$(S_{\langle\rangle})$ $\quad \langle\rangle \in S_\Delta$ *and* $\delta(\langle\rangle) = \Delta$

(S_{Ax}) *If* $s \in S_\Delta$ *and* $\delta(s)$ *is an axiom according to* (Ax L) *or* (Ax $\mathbb{N}$) *then* $s^\frown\langle i\rangle \notin S_\Delta$ *for all* $i \in \mathbb{N}$*. (I.e.* s *is a topmost node of* S_Δ*.)*

For the following clauses assume $s \in S_\Delta$ such that $\delta(s)$ is not an axiom.

(S_{id}) *If* $\delta(s)$ *is not reducible then* $s^\frown\langle 0\rangle \in S_\Delta$ *and* $\delta(s^\frown\langle 0\rangle) = \delta(s)$*.*

($S_{\bigwedge}$) *If* $\bigwedge\limits_{i\in I}\phi_i$ *is the distinguished formula in* $\delta(s)$ *then* $s^\frown\langle i\rangle \in S_\Delta$ *for all* $i \in I$ *and* $\delta(s^\frown\langle i\rangle) := \delta(s)^r, \phi_i$*.*

($S_{\bigvee}$) *Assume that* $\bigvee\limits_{i\in I}\phi_i$ *is the distinguished formula in* $\delta(s)$*. Let* i_0 *be the least* $i \in I$ *such that* ϕ_i *does not occur in* $\bigcup_{s_0\subseteq s}\delta(s_0)$*. Then* $s^\frown\langle i_0\rangle \in S_\Delta$ *and* $\delta(s^\frown\langle i_0\rangle) := \delta(s)^r, \phi_{i_0}, \bigvee\limits_{i\in I}\phi_i$*. If there is no such* i_0 *then* $s^\frown\langle 0\rangle \in S_\Delta$ *and* $\delta(s^\frown\langle 0\rangle) := \delta(s)^r$*.*

Observe that if we regard Definition 2.6 as an inductive definition we can dispense with clause (S_{Ax}). The least set which is closed under the remaining clauses satisfies (S_{Ax}) automatically.
For a function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ we define $f[n] := \langle f(0), \dots, f(n-1)\rangle$ and call $f[n]$ the course of values of f below n. If T is a tree and $f[m] \in T$ then $\{f[n] \mid n \le m\} \subseteq T$. We call $f[m]$ a path in the tree T. If $f[n] \in T$ for all $n \in \mathbb{N}$ we say that f is an infinite path in T. A tree is well–founded iff it has no infinite paths. For a well–founded tree S we denote by $\mathit{otyp}(S)$ its order–type, i.e., the ordinal measuring the depth of S.
A finite set Γ of formulas occurs in a path $f[m]$ in S_Δ if $\Gamma \subseteq \delta(f[n])$ for some $n \le m$. By "F occurs in a node $s \in S_\Delta$" we mean that F occurs in $\delta(s)$.
There are two main lemmas.

2.7 Lemma *(Syntactical Main-Lemma) If the search tree* S_Δ *is well-founded then* $\mathit{otyp}(S_\Delta) < \omega_1$ *and* $\overset{\mathrm{otyp}(S_\Delta)}{\models\!\!=} \Delta$ *.*

Proof: If the search tree S_Δ is well-founded then it is by definition countable. Hence $\mathit{otyp}(S_\Delta) < \omega_1$. If S_Δ is well–founded every topmost node of S_Δ contains an axiom and we get $\overset{\mathrm{otyp}(S_\Delta)}{\models\!\!=} \Delta$ easily by induction on $\mathit{otyp}(S_\Delta)$. □

2.8 Lemma *(Semantical Main-Lemma) If* S_Δ *is not well–founded then there is an assignment* $S_1, \dots, S_n$ *of subsets of* $\mathbb{N}$ *to the set variables in* Δ *such that* $\mathbb{N} \not\models \bigvee \Delta[S_1, \dots, S_n]$*.*

To sketch the proof we assume that f is an infinite path in S_Δ. We observe:

(0) *None of the* $\delta(f[n])$ *is an axiom.*

(1) *If A is an atomic formula occurring in some $s \in S_\Delta$ then A occurs in all t such that $s \subseteq t \in S_\Delta$.*

(2) *If a non atomic formula F occurs in some $f[n]$ then there is an $m \geq n$ such that F is distinguished in $f[m]$.*

The proof of (2) is an easy induction on the number of non atomic formulas occurring left of F in $\delta(f[n])$. Using *(2)* the proofs of the following observations are almost immediate from the definition of S_Δ.

(3) *If a formula $\bigwedge_{i \in I} \phi_i$ occurs in some $f[n]$ then there is an m and an $i \in I$ such that ϕ_i occurs in $f[m]$.*

(4) *If a formula $\bigvee_{i \in I} \phi_i$ occurs in $f[n]$ then there is for every $k \in I$ an m_k such that ϕ_k occurs in $f[m_k]$.*

To prove fact *(4)* we assume that $\bigvee_{i \in I} \phi_i$ is distinguished in $\delta(f[n])$. By *(2)* this means no loss of generality. If ϕ_k does not occur in $f[l]$ for $l \leq n$ then

$$\delta(f[n+1]) = \delta(f[n])^r, \phi_j, \bigvee_{i \in I} \phi_i$$

for some $j \in I$ such that $j \leq k$. Then ϕ_k will occur in $f[m+1]$ for some $m \geq n$ as soon as $\bigvee_{i \in I} \phi_i$ becomes distinguished in $\delta(f[m])$ and ϕ_l has occurred for all $l \in I \cap \{0, \ldots, k-1\}$.
We define an assignment

$$\Phi(X) := \{ t^{\mathbb{N}} \mid \ (t \notin X) \text{ occurs in } f \}.$$

Here φ occurs in f means that φ occurs in $f[n]$ for some n.
An easy induction on the length of a formula ψ, using observations *(3)* and *(4)* and the fact that f does not contain an axiom, shows

$$\mathbb{N} \not\models \psi[\Phi]$$

for all formulas ψ occurring in f. Since all formulas of Δ occur in $f[0]$ this yields the claim

$$\mathbb{N} \not\models \bigvee \Delta[\Phi]. \qquad \square$$

The Syntactical Main Lemma together with the Semantical Main Lemma prove Theorem 2.5. If $\phi \equiv \forall X_1 \ldots \forall X_n \psi(X_1, \ldots, X_n)$ and we assume $\not\models^{\alpha} \ \psi(X_1, \ldots, X_n)$ for all ordinals $\alpha < \omega_1$ then, by the Syntactical Main-Lemma, the search tree for $\psi(X_1, \ldots, X_n)$ is not well–founded. Applying the Semantical Main-Lemma we obtain an assignment Φ over $\mathbb{N}$ such that

$$\mathbb{N} \not\models \psi(X_1, \ldots, X_n)[\Phi].$$

Hence $\mathbb{N} \not\models \phi$. $\square$

If φ is a Π^1_1–sentence the search tree for φ^* as defined in Definition 2.6 is recursive. Therefore we can sharpen Theorem 2.5 to

2.9 Theorem *For a Π^1_1–sentence φ we have*

$$\mathbb{N} \models \varphi \;\Leftrightarrow\; (\exists \alpha < \omega_1^{\mathrm{CK}}) \models^{\alpha} \varphi^*.$$

There are two possibilites to relativize Theorem 2.9. Starting with a function $G\colon \mathbb{N} \longrightarrow \mathbb{N}$ we may introduce a constant for G. The computation of the value $t^{\mathbb{N}}$ of a term t in the extended language is primitive recursive in G. Starting with a set $S \subseteq \mathbb{N}$ we may introduce a constant for S. In both cases Diagram($\mathbb{N}$) becomes recursive in G or S, respectively. Therefore the search tree for a Π^1_1–formula in the extended language becomes recursive in G or S. Putting

$$\omega_1^{\mathrm{CK}}[S] := \sup \{ \mathit{otyp}(\prec) \mid\; \prec \text{ is recursive in } S \}$$

we get

2.10 Theorem *For a Π^1_1–sentence $\varphi(S)$ with parameter S we have*

$$\mathbb{N} \models \varphi(S) \;\Leftrightarrow\; (\exists \alpha < \omega_1^{\mathrm{CK}}[S]) \models^{\alpha} \varphi^*(S).$$

As a side remark one should notice that the Hyperarithmetical Quantifier Theorem follows from Theorem 2.10. Since the search tree for $\varphi^*(S)$ can easily be constructed within $L_{\omega_1^{\mathrm{CK}}[S]}(S)$ which is *Hyp*(S) we get

2.11 Theorem *For any Π^1_1–formula $\varphi(\vec{x}, \vec{X})$ there is a Σ_1–formula $\psi(\vec{x}, \vec{X})$ in the language of set theory such that*

$$(\forall \vec{S} \subseteq \omega)(\forall \vec{n} \in \omega^k) \left[\mathbb{N} \models \varphi[\vec{n}, \vec{S}] \;\Leftrightarrow\; L_{\omega_1^{CK}[\vec{S}]}(\vec{S}) \models \psi[\vec{n}, \vec{S}] \right].$$

Theorem 2.10 can be extended to sentences which are positively arithmetical in Π^1_1–sets, i.e., to sentences of the form $\phi(\psi_1, \ldots, \psi_n)$ where $\phi(X_1, \ldots, X_n)$ is an $X_1, \ldots, X_n$–positive arithmetical formula and $\psi_1, \ldots, \psi_n$ are Π^1_1–formulas. The proof needs a lemma saying

$$\models^{\alpha} \exists x \phi(x, (X)_x) \;\Rightarrow\; \models^{\alpha \cdot 2} \exists x \forall X \phi(x, X)$$

where $(X)_x := \{ z \mid \langle z, x \rangle \in X \}$. A proof is in [2].

3 The boundedness theorem

Using Theorem 2.5 we define for $\mathcal{L}_2^{\infty}$–sentences φ

$$\mathit{tc}(\varphi) := \begin{cases} \min \left\{ \alpha \mid \models^{\alpha} \varphi \right\} & \text{if this exists} \\ \omega_1 & \text{otherwise.} \end{cases}$$

We call $\mathsf{tc}(\varphi)$ the truth–complexity of φ which is motivated by the fact that for first order sentences $\models^{\alpha} \varphi$ is just the truth definition for φ. For Π^1_1–sentences φ of $\mathcal{L}_2$ we define

$$\mathsf{tc}(\varphi) := \mathsf{tc}(\varphi^*).$$

Let

$$\mathrm{Prog}(\prec, X) :\equiv \forall x[\forall y(y \prec x \to y \in X) \to x \in X]$$

and

$$TI(\prec) :\equiv \forall X[\mathrm{Prog}(\prec, X) \to \forall x(x \in X)].$$

Then $TI(\prec)$ expresses transfinite induction along $\prec$ and for arithmetical definable $\prec$ the sentence $TI(\prec)$ is Π^1_1. We will see that there is a close connection between the truth complexity of the sentence $TI(\prec)$ and the order–type $\mathsf{otyp}(\prec)$ of the well–ordering $\prec$. First we observe that there is a canonical infinitary proof for $TI(\prec)$. Let $\prec$ be a – for simplicity primitive recursive – well-founded binary relation. We show

$$\models\!\!=\!\!=^{5\cdot(otyp_{\prec}(n)+1)} \neg\mathrm{Prog}(\prec, X), \underline{n} \in X \tag{6}$$

by induction on $\mathsf{otyp}_{\prec}(n)$. We have

$$\models\!\!=\!\!=^{5\cdot(otyp_{\prec}(m)+1)} \neg\mathrm{Prog}(\prec, X), \neg\underline{m} \prec \underline{n}, \underline{m} \in X \tag{i}$$

either as an instance of (Ax $\mathbb{N}$) or by induction hypothesis. Hence

$$\models\!\!=\!\!=^{5\cdot otyp_{\prec}(n)+3} \neg\mathrm{Prog}(\prec, X), (\forall y)[y \prec \underline{n} \to y \in X] \tag{ii}$$

by two applications of ($\bigvee$) and one application of ($\bigwedge$). By (AxL) we have

$$\models^{0} \neg\mathrm{Prog}(\prec, X), \underline{n} \notin X, \underline{n} \in X\ . \tag{iii}$$

By (ii) and (iii) we obtain

$$\models\!\!=\!\!=^{5\cdot otyp_{\prec}(n)+4} \neg\mathrm{Prog}(\prec, X), (\forall y)[y \prec \underline{n} \to y \in X] \land \underline{n} \notin X, \underline{n} \in X\ . \tag{iv}$$

One additional “inference” ($\bigvee$) leads to

$$\models\!\!=\!\!=^{5\cdot(otyp_{\prec}(n)+1)} \neg\mathrm{Prog}(\prec, X), \underline{n} \in X\ . \qquad \square$$

From (6) we obtain by a clause ($\bigwedge$), two clauses ($\bigvee$) and one application of ($\forall$) the following lemma.

3.1 Lemma *If $\prec$ is a primitive recursive well-founded relation whose order-type is a limit ordinal then* $\mathsf{tc}(TI(\prec)) \leq \mathsf{otyp}(\prec) + 3$.

The claim in Lemma 3.1 should of course be read as

$$\mathit{tc}(TI(\prec)) \leq \mathit{otyp}(\prec) \tag{7}$$

since the "+3" is due to the technical peculiarities of the calculus. Observe that the Lemma remains true if we replace $\prec$ by an arithmetical definable well–ordering whose order-type is a limit. Instead of using (Ax $\mathbb{N}$) in (i) we have to use that a true arithmetical sentence ψ has truth complexity $\leq 2 \cdot \mathsf{rk}(\psi)$.
We are going to show that we also have the converse inequality. In [5] Theorem 13.10 there is a proof of $\mathit{otyp}(\prec) \leq 2^{\mathit{tc}(TI(\prec))}$ — a result which goes back to Gentzen. Quite recently A. Beckmann has improved that to $\mathit{otyp}(\prec) \leq \mathit{tc}(TI(\prec))$ (cf. [1]) which turned out to be important for the proof theory of certain subsystems of arithmetic. Though not really essential for the results of this paper we want to prove the sharper version. We need some notations. Recall that every class $\mathcal{O} \subseteq \mathit{On}$ has a uniquely determined enumerating function

$$\mathit{en}_{\mathcal{O}} \colon \mathit{On} \longrightarrow_{\mathrm{p}} \mathcal{O}$$

which is recursively defined by $\mathit{en}_{\mathcal{O}}(\alpha) = \min\{\xi \in \mathcal{O} \mid (\forall \beta < \alpha)[\mathit{en}_{\mathcal{O}}(\beta) < \xi]\}$. Observe that $\mathit{en}_{\mathcal{O}}$ is partial iff $\mathcal{O}$ is a set. We introduce also the dual enumerating function $\overline{\mathit{en}}_{\mathcal{O}} := \mathit{en}_{\mathrm{On}\setminus\mathcal{O}}$ which enumerates the complement of $\mathcal{O}$. For the dual enumerating function we obviously have

$$A \subseteq B \;\; \Rightarrow \;\; \forall \alpha(\overline{\mathit{en}}_A(\alpha) \leq \overline{\mathit{en}}_B(\alpha)).$$

Let $\prec$ be an order–relation. Its accessible part can be inductively defined by the monotone operator

$$\mathsf{A}_{\prec}(S) := S \cup \{n \in \mathbb{N} \mid \forall m(m \prec n \rightarrow m \in S)\}.$$

Defining the α–th iteration of this operator as

$$\mathsf{A}^{\alpha}_{\prec}(S) := \mathsf{A}_{\prec}(S \cup \bigcup_{\xi<\alpha} \mathsf{A}^{\xi}_{\prec}(S))$$

we obtain the α–th stage of the inductive definition as

$$\mathsf{A}^{\alpha}_{\prec} := \mathsf{A}^{\alpha}_{\prec}(\emptyset).$$

By $\mathit{Acc}(\prec) := \bigcup_{\xi \in \mathrm{On}} \mathsf{A}^{\xi}_{\prec}$ we denote the accessible part of $\prec$.
For $n \in \mathit{Acc}(\prec)$ its order–type is obtained as

$$\mathit{otyp}_{\prec}(n) = \min\{\alpha \mid n \in \mathsf{A}^{\alpha}_{\prec}\} \tag{8}$$

and we obtain

$$\mathit{otyp}(\prec) := \sup\{\mathit{otyp}_{\prec}(x) + 1 \mid x \in \mathit{Acc}(\prec)\}.$$

We are going to modify the accessibility operator. For $M \subseteq \mathbb{N}$ let

$$M^{\prec} := \{\mathit{otyp}_{\prec}(m) \mid m \in M\}$$

and define

$$\mathsf{R}^\alpha_\prec(M) := \{n \in \mathbb{N} \mid \textsf{\textit{otyp}}_\prec(n) \le \overline{\mathrm{en}}_{M^\prec}(\alpha)\} \cup M.$$

Putting

$$\mathsf{R}^{<\alpha}_\prec(M) := \bigcup_{\beta < \alpha} \mathsf{R}^\beta_\prec(M)$$

we get

$$\mathsf{A}_\prec\left(M \cup \mathsf{R}^{<\alpha}_\prec(M)\right) \subseteq \mathsf{R}^\alpha_\prec(M). \tag{9}$$

In proving (9) we first observe that both sides contain the set M. To show the inclusion from left to right assume $n \notin M$ and $n \in \mathsf{A}_\prec\left(M \cup \mathsf{R}^{<\alpha}_\prec(M)\right)$. Then

$$(\forall m \prec n)[(\exists \beta < \alpha)\textsf{\textit{otyp}}_\prec(m) \le \overline{\mathrm{en}}_{M^\prec}(\beta) \vee m \in M]$$

which entails $\textsf{\textit{otyp}}_\prec(m) < \overline{\mathrm{en}}_{M^\prec}(\alpha)$ for all $m \prec n$. Hence $\textsf{\textit{otyp}}_\prec(n) \le \overline{\mathrm{en}}_{M^\prec}(\alpha)$ which shows $n \in \mathsf{R}^\alpha_\prec(M)$. The opposite inlusion of (9) is only true if M fulfills $\forall n(\textsf{\textit{otyp}}_\prec(m) \in M \to m \in M)$. To prove this assume $\textsf{\textit{otyp}}_\prec(n) \le \overline{\mathrm{en}}_{M^\prec}(\alpha)$. Let $m \prec n$. If $\overline{\mathrm{en}}_{M^\prec}(\beta) < \textsf{\textit{otyp}}_\prec(m) < \textsf{\textit{otyp}}_\prec(n) \le \overline{\mathrm{en}}_{M^\prec}(\alpha)$ for all $\beta < \alpha$ then $m \in M$. Otherwise we have $\textsf{\textit{otyp}}_\prec(m) \le \overline{\mathrm{en}}_{M^\prec}(\beta)$ for some $\beta < \alpha$. Hence $(\forall m \prec n)[m \in M \cup \mathsf{R}^{<\alpha}_\prec(M)]$ which implies $n \in \mathsf{A}_\prec(M \cup \mathsf{R}^{<\alpha}_\prec(M))$.
From (9) we get by induction on α

$$\mathsf{A}^\alpha_\prec(M) \subseteq \mathsf{R}^\alpha_\prec(M). \tag{10}$$

From the obvious fact

$$\overline{\mathrm{en}}_{M^\prec \cup \{\mathrm{otyp}(n)\}}(\alpha) \le \overline{\mathrm{en}}_{M^\prec}(\alpha + 1)$$

we get

$$\mathsf{R}^\alpha_\prec(M \cup \{n\}) \subseteq \mathsf{R}^{\alpha+1}_\prec(M) \cup \{n\}. \tag{11}$$

3.2 Lemma *(Boundedness Lemma) Let $\prec_{\vec{Y}}$ be a binary relation which is definable by an $\mathcal{L}_2$ formula and $X, Y_1, \ldots, Y_m$ be a list of set variables containing all the variables occurring in Δ and $\prec_{\vec{Y}}$. Assume that X does not occur among the variables in the defining formula for $\prec_{\vec{Y}}$ and occurs only positively in Δ, i.e., there are no occurrences $s \notin X$ in Δ. If*

$$\models^{\alpha} \neg\mathrm{Prog}(\prec_{\vec{Y}}, X), s_1 \notin X, \ldots, s_n \notin X, \Delta \tag{12}$$

then

$$\mathbb{N} \models \bigvee \Delta[\mathsf{R}^\alpha_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\}), S_1, \ldots, S_m]$$

holds for all sets $S_i \subseteq \mathbb{N}$, $i = 1, \ldots, m$ such that $\prec_{\vec{S}}$ is a well–ordering.

Proof: We induct on α and show only the two interesting cases. If (12) holds according to an axiom (Ax L) and there is a formula $t \in X$ in Δ such that $t^{\mathbb{N}} = s_i^{\mathbb{N}}$ for some $i \in \{1, \ldots, n\}$ then $t^{\mathbb{N}} \in \mathsf{R}^\alpha_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\})$ and we are done.

If (12) holds according to an inference ($\bigvee$) whose derived formula is $\neg\mathrm{Prog}(\prec_{\vec{Y}}, X)$ we have the premise

$$\models^{\alpha_0} \neg\mathrm{Prog}(\prec_{\vec{Y}}, X), \forall x(x \prec_{\vec{Y}} s \to x \in X) \wedge s \notin X, s_1 \notin X, \ldots, s_n \notin X, \Delta \quad \text{(i)}$$

for some $\alpha_0 < \alpha$ and some term s. By $\bigwedge$–inversion we obtain from (i)

$$\models^{\alpha_0} \neg\mathrm{Prog}(\prec_{\vec{Y}}, X), \forall x(x \prec_{\vec{Y}} s \to x \in X), s_1 \notin X, \ldots, s_n \notin X, \Delta \quad \text{(ii)}$$

as well as

$$\models^{\alpha_0} \neg\mathrm{Prog}(\prec_{\vec{Y}}, X), s \notin X, s_1 \notin X, \ldots, s_n \notin X, \Delta. \quad \text{(iii)}$$

If there is some $k \prec_{\vec{S}} s^{\mathbb{N}}$ such that $k \notin \mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\})$ then we get

$$\mathbb{N} \models \bigvee \Delta[\mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\}), S_1, \ldots, S_m] \quad \text{(iv)}$$

from (ii) by the induction hypothesis. Since X occurs only positively in Δ and $\mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\}) \subseteq \mathsf{R}^{\alpha}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\})$ the claim follows from (iv).
If $k \in \mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\})]$ for all $k \prec_{\vec{S}} s^{\mathbb{N}}$ we get

$$s^{\mathbb{N}} \in \mathsf{R}^{\alpha_0+1}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\}) \quad \text{(v)}$$

by (10). The induction hypothesis for (iii) yields

$$\mathbb{N} \models \bigvee \Delta[\mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}, s^{\mathbb{N}}\}), S_1, \ldots, S_m]. \quad \text{(vi)}$$

By (11) and (v) we have

$$\begin{aligned} \mathsf{R}^{\alpha_0}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}, s^{\mathbb{N}}\}) &\subseteq \mathsf{R}^{\alpha_0+1}_{\prec_{\vec{S}}}(\{s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}\}) \cup \{s^{\mathbb{N}}\} \\ &\subseteq \mathsf{R}^{\alpha}_{\prec_{\vec{S}}}(s_1^{\mathbb{N}}, \ldots, s_n^{\mathbb{N}}) \end{aligned} \quad \text{(vii)}$$

and the claim follows from (vi) and (vii) and the fact that X occurs only positively in Δ.
The remaining cases are either trivial or follow straight forwardly from the induction hypothesis. □

3.3 Theorem *(Boundedness Theorem) For a well–ordering $\prec$ of $\mathbb{N}$ we have*

$$\mathit{otyp}(\prec) \leq \mathit{tc}(TI(\prec)).$$

Proof: For $\alpha := \mathit{tc}(TI(\prec))$ we have

$$\models^{\alpha} \forall X[\mathrm{Prog}(\prec, X) \;\; \to \;\; \forall x(x \in X)]$$

and obtain an $\alpha_0 < \alpha$ such that

$$\models^{\alpha_0} \neg\mathrm{Prog}(\prec, X), \forall x(x \in X).$$

Hence $\forall n(n \in \mathsf{R}^{\alpha_0}_{\prec}(\emptyset))$ by the Boundedness Lemma which entails

$$\mathit{otyp}(\prec) = \sup\{\mathit{otyp}_{\prec}(n)+1 \mid n \in \mathbb{N}\} \leq \alpha.$$

$\square$

As an immediate consequence of Theorem 2.9 and the Boundedness Theorem we get

3.4 Corollary *We have* $\mathit{otyp}(\prec) < \omega_1^{\mathrm{CK}}$ *for all arithmetically definable well–orderings* $\prec$.

To generalize Corollary 3.4 we introduce the following notations.
Let

$$n \prec_R m \ :\Leftrightarrow\ \langle m, n\rangle \in R$$

and

$$\mathbb{W} := \{R \subseteq \mathbb{N} \mid \ \prec_R \text{ is a well–ordering}\}.$$

To every index e of a computable function we associate the binary relation

$$x \prec_e y \ :\Leftrightarrow\ \{e\}(\langle x, y\rangle) = 0.$$

Let

$$\mathsf{W} := \{e \mid \ \{e\} \text{ is total} \wedge \ \prec_e \text{ is a well–ordering}\}.$$

3.5 Theorem *(Boundedness Principle) If* P *is a* Σ^1_1*–definable subclass of* $\mathbb{W}$ *then* $\sup\{\mathit{otyp}(\prec_R) \mid R \in P\} < \omega_1^{\mathrm{CK}}$.

Proof: Assume

$$\mathbb{N} \models \forall Y[Y \in P \to TI(\prec_Y)] \tag{i}$$

and assume that P is defined by a Σ^1_1–formula, say $P = \{Y \mid \exists Z F(Z, Y)\}$. Then

$$\mathbb{N} \models \forall Y \forall Z \forall X[\neg\mathrm{Prog}(\prec_Y, X) \vee \neg F(Z, Y) \vee \forall x(x \in X)]. \tag{ii}$$

The formula in (ii) is Π^1_1. Therefore there is an $\alpha < \omega_1^{\mathrm{CK}}$ such that

$$\models^{\!\!\!\alpha}\ \neg\,\mathrm{Prog}(\prec_Y, X), \neg F(Z, Y), \forall x(x \in X). \tag{iii}$$

Let $R \in P$. Then $\prec_R$ is a well–ordering and there is some $S \subseteq \mathbb{N}$ such that $F(S, R)$. By the Boundedness Lemma we get

$$\mathbb{N} \models \neg F(S, R) \vee \forall x[x \in \mathsf{R}^{\alpha}_{\prec_R}(\emptyset)] \tag{iv}$$

which shows that $\mathit{otyp}(\prec_R) \leq \alpha + 1$. Therefore we have

$$\sup\{\mathit{otyp}(\prec_R) \mid R \in P\} \leq \alpha + 1 < \omega_1^{\mathrm{CK}}. \qquad \square$$

Completely analogous we obtain also

3.6 Theorem *If P is a Σ^1_1–definable subclass of* W *then*

$$\sup\{\mathsf{otyp}(\prec_e)\mid\ e \in P\} < \omega_1^{\text{CK}}.$$

As a corollary of Theorem 3.5 we obtain

3.7 Theorem *If $\prec$ is a Σ^1_1–definable well–ordering then $\mathsf{otyp}(\prec) < \omega_1^{\text{CK}}$.*

Proof: Define

$$P = \{X \mid\ \{(x,y) \mid\ \langle x,y\rangle \in X\} \text{ is a linear order } \wedge \forall x \forall y[\langle x,y\rangle \in X \to x \prec y]\}.$$

Then P is a Σ^1_1–definable subclass of $\mathbb{W}$. By Theorem 3.5 we get

$$\sup\{\mathsf{otyp}(\prec_X)\mid\ X \in P\} < \omega_1^{\text{CK}}.$$

From the definition of P, however, it is obvious that

$$\mathsf{otyp}(\prec) = \sup\{\mathsf{otyp}(\prec_X)\mid\ X \in P\}. \qquad \Box$$

Relativizing Theorems 3.6 and 3.7 we get

3.8 Theorem *Let P be a $\Sigma^1_1[G]$–definable subclass of $\mathbb{W}$ then*

$$\sup\{\mathsf{otyp}(\prec_F)\mid\ F \in P\} < \omega_1^{\text{CK}}[G].$$

3.9 Theorem *If $\prec$ is a $\Sigma^1_1[G]$–definable well–ordering then $\mathsf{otyp}(\prec) < \omega_1^{\text{CK}}[G]$.*

Theorem 3.8 entails also its boldface version

3.10 Theorem *Let P be a $\boldsymbol{\Sigma^1_1}$–definable subclass of $\mathbb{W}$ then*

$$\sup\{\mathsf{otyp}(\prec_F)\mid\ F \in P\} < \omega_1.$$

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